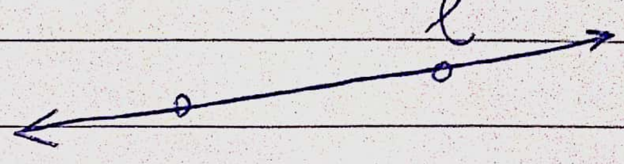
MTH 201 | 22 AUG 22

Curves & Surfaces

Recall

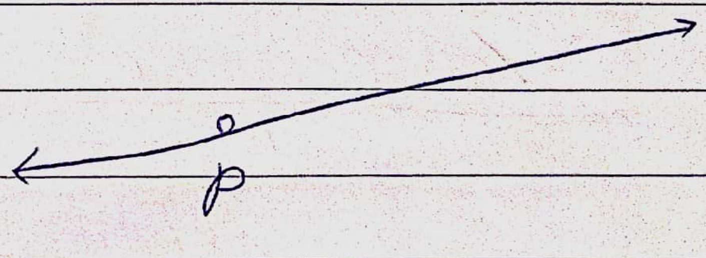
(Geometry)



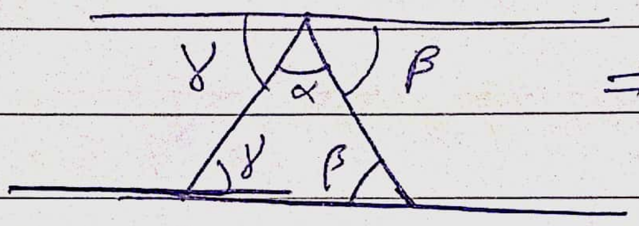
Main axioms: ① line can be extended

② parallel lines

∃! line through p parallel to l.



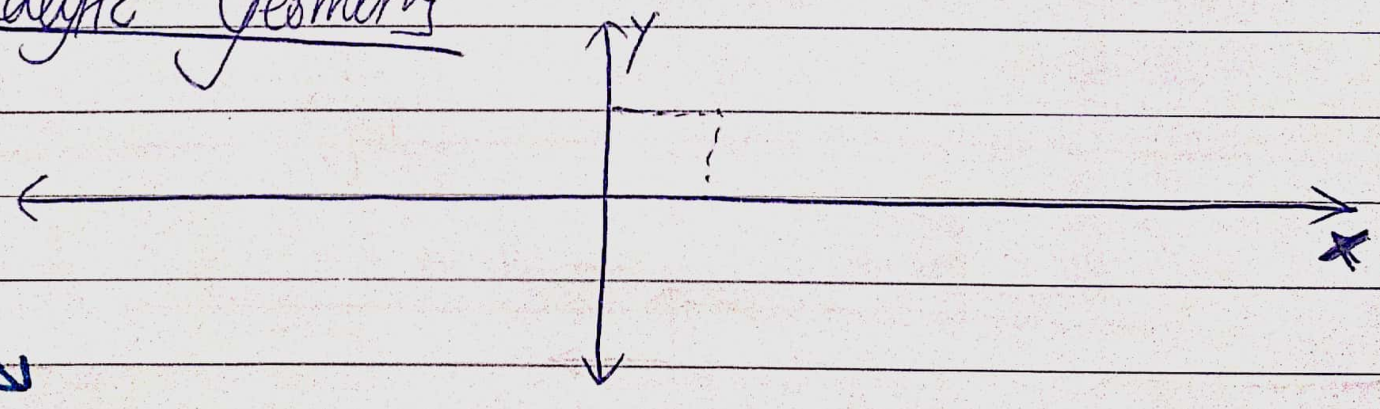
Main results



$\Rightarrow \alpha + \beta + \gamma = 180°$

Congruency of triangles ⟶ only quantify angles but also "measure" lines?

Analytic Geometry

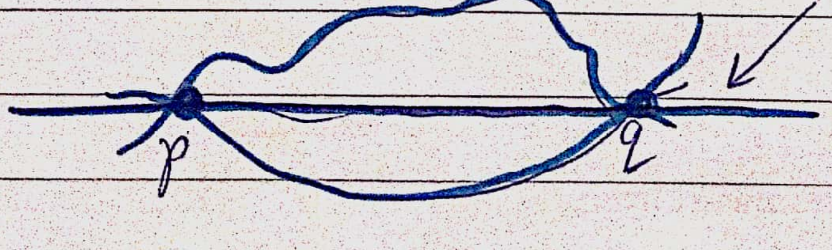


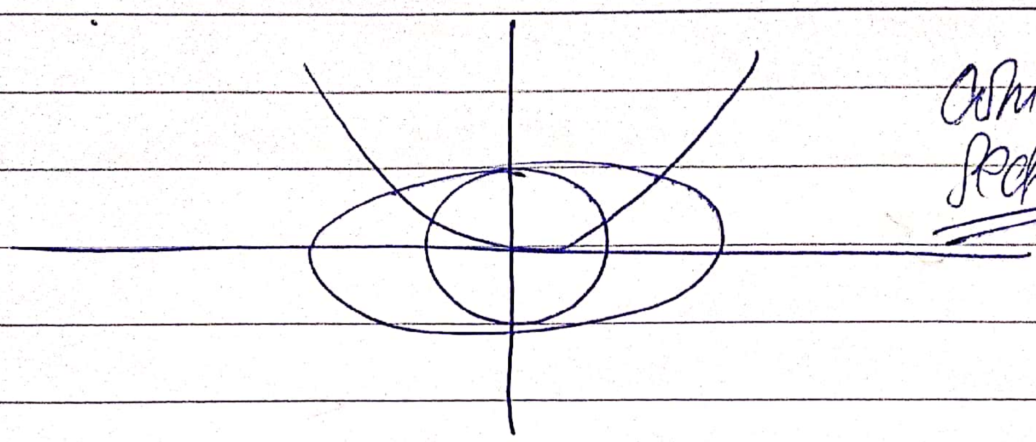
metric space ↘

$(\mathbb{R}^2, d)$ where $d\left(\begin{matrix}x_1 \\ y_1\end{matrix}, \begin{matrix}x_2 \\ x_2\end{matrix}\right) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

Shortest distance path is the line! → w/ dist. $= d(p, q)$

line.





Conic Sections

$\boxed{\text{Vector Space}}$ $(V, +, \cdot)$

Take $V = \mathbb{R} \times \mathbb{R}$

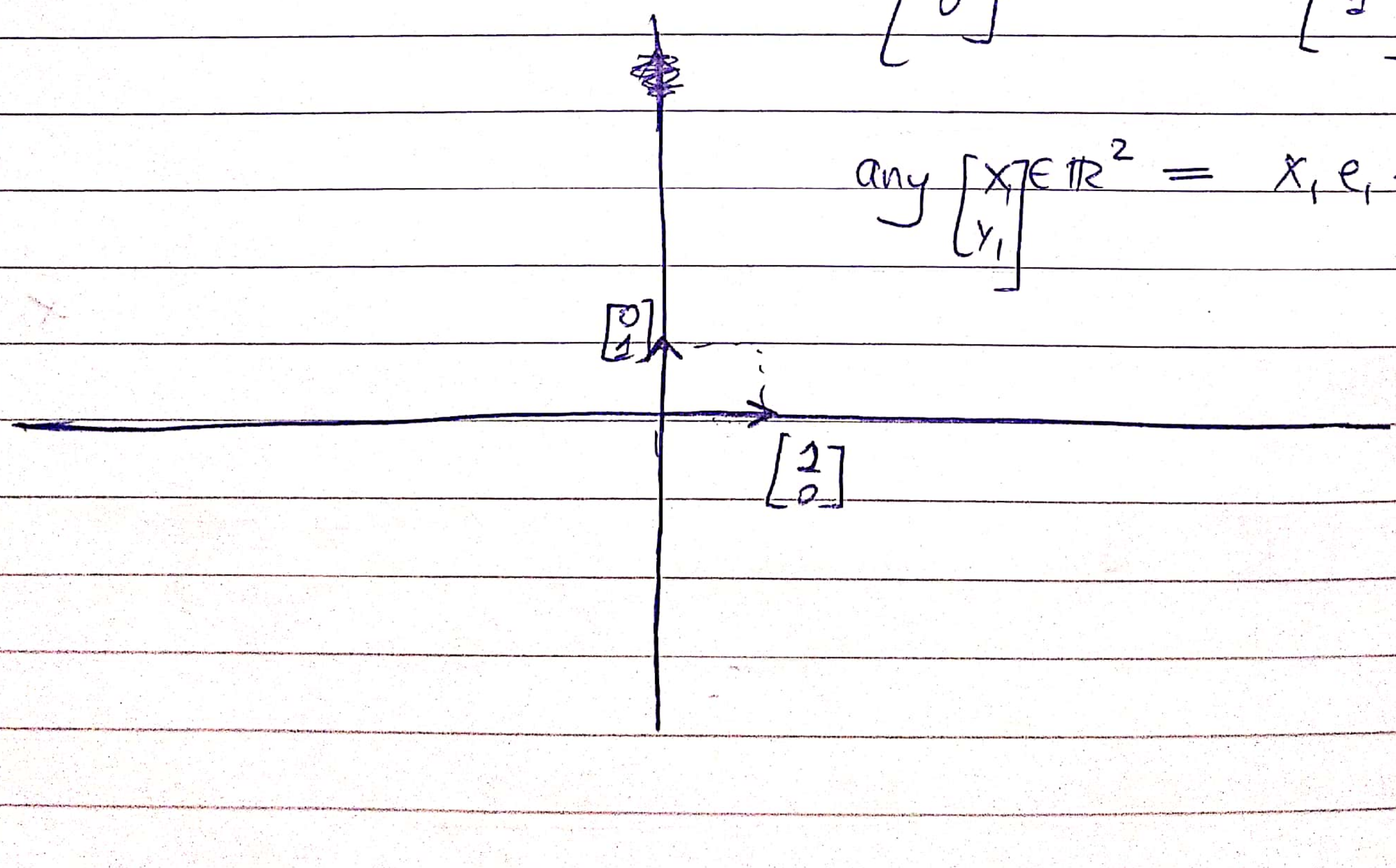
$$\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} := \begin{bmatrix} a+c \\ b+d \end{bmatrix} \text{ in } \mathbb{R}.$$

$$\alpha \begin{bmatrix} a \\ b \end{bmatrix} := \begin{bmatrix} \alpha a \\ \alpha b \end{bmatrix}$$

$\mathbb{R}^2 \rightsquigarrow$ also ~~a~~ has a basis

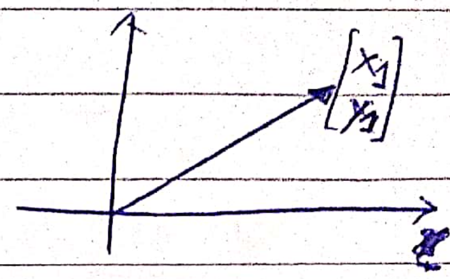
$\hookrightarrow$ natural basis $\to$ $e_1 := \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

any $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \in \mathbb{R}^2 = x_1 e_1 + x_2 e_2$.

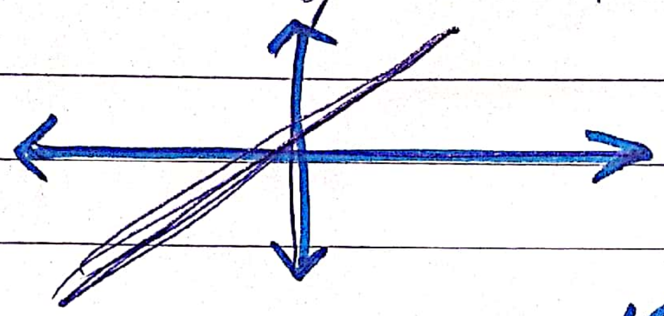


In $\mathbb{R}^3$, we live.

In vector spaces, origin is special.



Vector subspaces of $\mathbb{R}^2$



are lines passing through origin.

But We remove the speciality of $\begin{bmatrix}0\\0\end{bmatrix}$

by associating rigid motion $x \longmapsto A\vec{x} + b$

linear — some $\in \mathbb{R}^2$.

isometries of $d=\sqrt{(\Delta x)^2+(\Delta y)^2}$

We generalize/change to affine space.

we do geometry there

Irony: ~~Classical mechanics~~ is used to start geometry

$(\mathbb{R}^2, d)$

$f:\mathbb{R}^2\to\mathbb{R}$ is (a isometry)

$d(f(x), f(y)) = d(x,y)$.

$\Longrightarrow$

$\boxed{f \text{ is affine!}}$

$f(\vec{x}) = O(\vec{x}) + b$

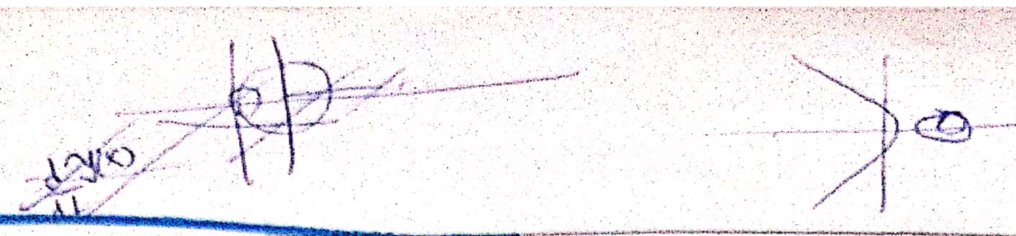
$OO^T = I$ — any $b \in \mathbb{R}^2$

rigid linear

Hence, $(\mathbb{R}^2, d_2)$ is our model for Geometry.

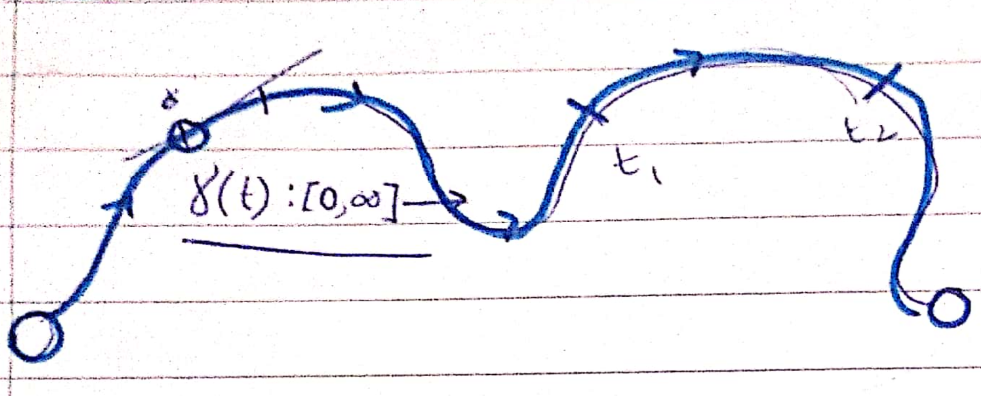
I prefer the vector space with the distance function structure.

We can take an affine space.



$x^3 - x + 1 = 0$

✓ Measuring lengths of curves



curve γ

magnitude of velocity

$$\underbrace{\int_{[t_1,t_2]} \|\gamma'(t)\| \, dt}_{\text{"the distance covered"} \;=\; \text{length}}$$

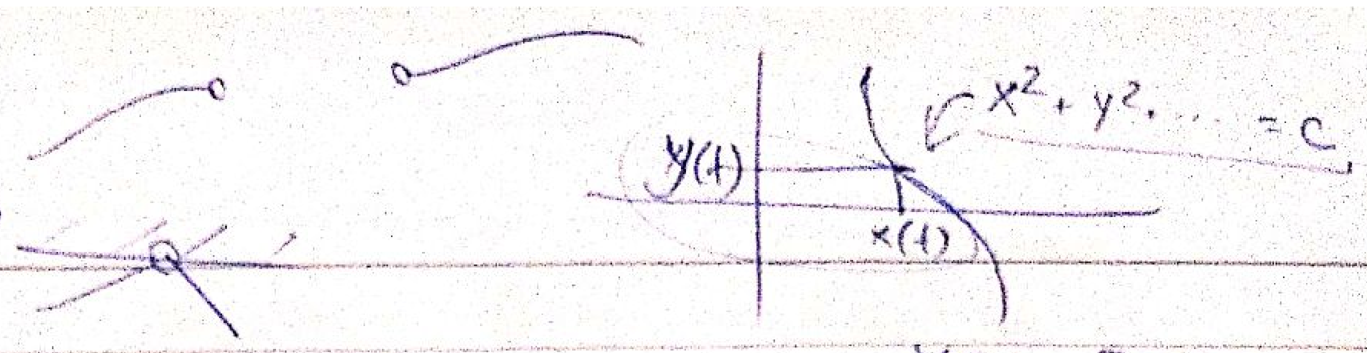
$$\text{length}(\gamma) := \int_{[0,\infty]} \|\gamma'(t)\|_2 \, dt$$

$\boxed{\text{Textbook}}$ { Christian Bär
Monfredo De Carlo }

$\boxed{\text{Aim}}$: Learn $\boxed{\text{geometry}}$ of curves
$\boxed{\text{Geometry}}$ of surfaces

→ lengths, areas,
curvatures

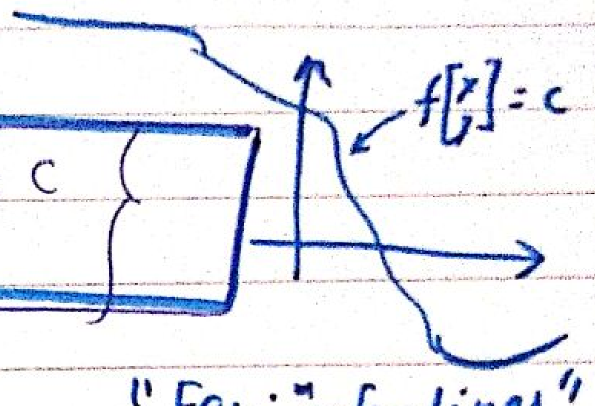
Formally:



Formally, a curve α is given by $\qquad \gamma(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$

$$f\begin{bmatrix}x\\y\end{bmatrix} = c.$$

$$\boxed{\alpha := \left\{ \begin{bmatrix}x\\y\end{bmatrix} \in \mathbb{R}^2 \;\middle|\; f\begin{bmatrix}x\\y\end{bmatrix} = c \right\}}$$



"Equi-f-lines"

Suppose we have $\gamma : I \longrightarrow \mathbb{R}^2$, $I = [a,b]$

this is parameter...

(Defn) Let $I = [a,b]$ be an closed interval in $\mathbb{R}$.

A $\boxed{\text{parameterized curve}}$ is a map

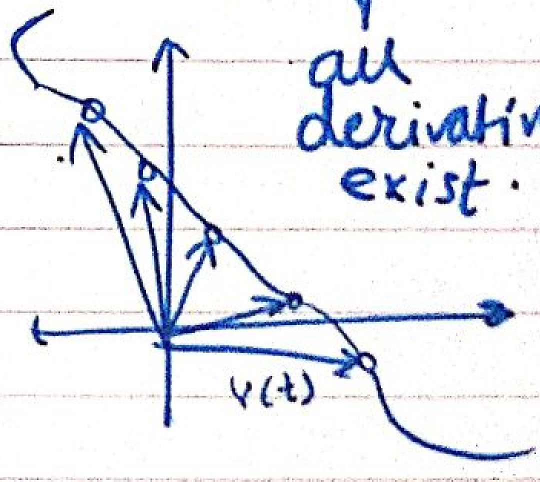
$c : I \longrightarrow \mathbb{R}^2$ such that $\qquad$ cont.-diff.

$c(t) \in C^\infty(I, \mathbb{R}^2)$

all derivatives exist.

$$c = \begin{bmatrix} \gamma_1(t) \\ \gamma_2(t) \end{bmatrix} \qquad c'(t) := \begin{bmatrix} \gamma_1'(t) \\ \gamma_2'(t) \end{bmatrix}$$

"$\left(\frac{d}{dt} c(t)\right)$"



"position vector γ as a fcn of t"

parameter

$$c''(t) := \begin{bmatrix} \gamma_1''(t) \\ \gamma_2''(t) \end{bmatrix}$$

and so on.

all derivatives exist.

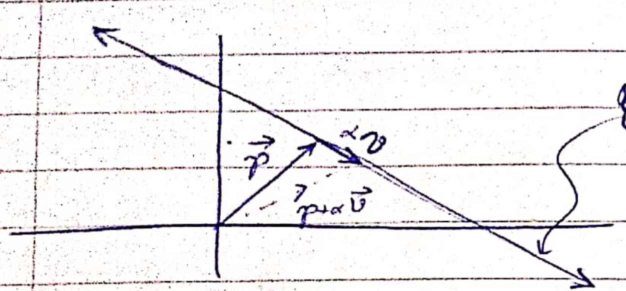
$\boxed{\text{A curve } c(t) \text{ is } \underline{\text{smooth}} \text{ if } c \in \mathcal{C}^\infty}$

st. line

$\boxed{\text{Example}}$ $\gamma(t) : \mathbb{R} \longrightarrow \mathbb{R}^2$

$\gamma(t) := \vec{p} + t\vec{v}$ $\qquad p \in \mathbb{R}^2$

$v \in \mathbb{R}^2 - \left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}$



$\boxed{\text{We note} \quad \gamma(t) \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^2)}$

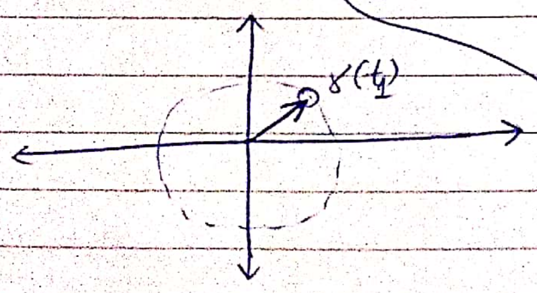
$\boxed{\text{Defn}}$ A par. curve γ is called $\underline{\text{regular}}$ if

$$\gamma'(t) \neq 0 \qquad \forall\, t \in I.$$

<u>Noted</u>: <u>St. line is a regular curve</u>.

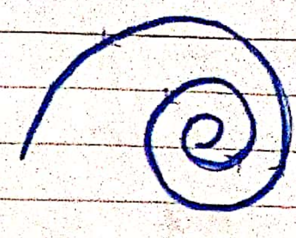
$\boxed{\text{Example}}$ $\gamma := \begin{bmatrix} r\cos(t) \\ r\sin(t) \end{bmatrix}$, $\quad r > 0, \in \mathbb{R}$

$t \in [0, 2\pi]$



γ is <u>regular</u>.

$\boxed{\text{Example}}$ $c(t) := e^{t/10} \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$



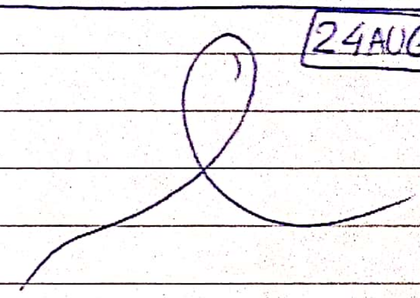
<u>logarithmic spiral</u>

$*$ $\boxed{\text{Next}}$ $\Rightarrow$ change of parametrization; makes lives easy?

does not depends on parametrization

Quiz/Assign.	$-$ 20%
Midsem	$-$ 20% + 20%
Endsem	$-$ 40%

$\boxed{\text{24AUG22}}$



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$f : [a, b] \longrightarrow \mathbb{R}^3$

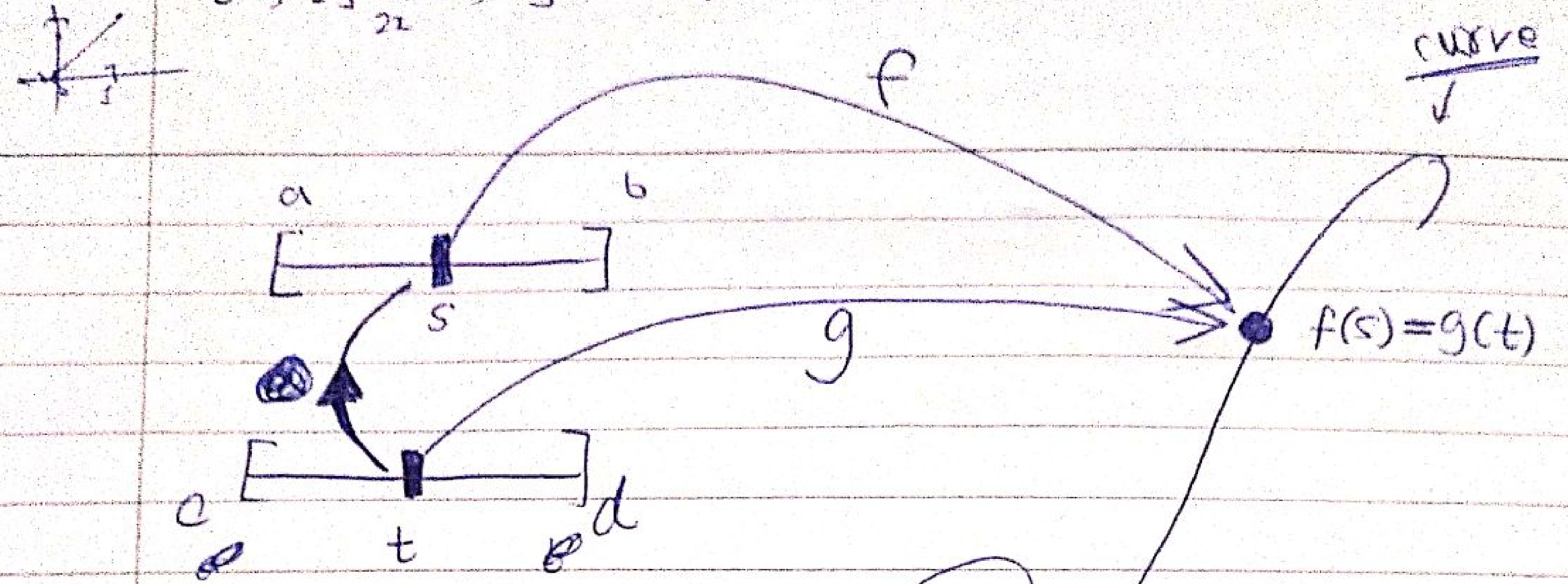
$f : [0, \pi] \longrightarrow \mathbb{R}^2 \longrightarrow$ lengths for both curves must be same!

$f(t) := \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$

$g : [0, \frac{1}{2}] \longrightarrow \mathbb{R}^2$

$g(t) := \begin{bmatrix} \cos 2\pi t \\ \sin 2\pi t \end{bmatrix}$

$[0,1] \xleftrightarrow{2x} [0,2] \longleftrightarrow \mathbb{R} \longleftrightarrow \mathbb{R}^3$



(Defn) Given smooth parm. $f : [a,b] \longrightarrow \mathbb{R}^3$

Say $\exists\ \theta : [c,d] \longrightarrow [a,b]$

$\hookrightarrow$ smooth, bijective.

s.t. $g := f \circ \theta$

$g : [c,d] \longrightarrow \mathbb{R}^3$ is smooth.

then $\theta \longleftarrow$ coordinate transfn

$g \longleftarrow$ a reparam. of f.

(Defn) A unit speed parameterization $\gamma : [a,b] \longrightarrow \mathbb{R}^3$ is a parameterization such that

$$\|\dot{\gamma}(t)\| = 1, \quad \forall\ t \in [a,b].$$

curvature $\longleftrightarrow \dfrac{d\vec{v}}{dt}$

26AUG22

(Def) A parameterization $\gamma : [a,b] \longrightarrow \mathbb{R}^3$ is regular if $\gamma'(t) \neq 0$ for any $t \in [a,b]$.

(Thm) Any regular parametrization has a unit speed reparameterization

Take $\gamma(t) : [a,b] \longrightarrow \mathbb{R}^3$
as $\gamma(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$

Hence, $\gamma'(t) = \begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix}$

Say $\gamma_1 : [a,b] \longrightarrow \mathbb{R}^3$

$\theta : [c,d] \longrightarrow [a,b]$

now define

$\gamma_2 := \gamma_1 \circ \theta$
where
$\gamma_2 : [c,d] \longrightarrow \mathbb{R}^3$

$$\gamma_2(t) = (\gamma_1 \circ \theta)(t)$$

$$\gamma_2'(t) = \gamma_1' \circ \theta(t) \times \theta'(t)$$

$$= \begin{bmatrix} x_1'(\theta(t)) \\ y_1'(\theta(t)) \\ z_1'(\theta(t)) \end{bmatrix} \times \theta'(t) \Rightarrow \gamma_2' = (\gamma_1' \circ \theta) \times \theta'$$

$$\|\gamma_2'\| = \|\gamma_1' \circ \theta\| \, \theta'(t) = 1 \quad \text{(to happen)}$$

$$\Rightarrow \quad \theta'(t) = \frac{1}{\|\gamma_1' \circ \theta\|} \quad \text{(want)}$$

Only possible iff $\gamma_1'(t) \neq 0 \quad \forall \; t \in [a,b]$.

I

(Fundamental Theorem of Calculus)

$$F(x) := \int_{[a,x]} f \quad \text{then} \quad F'(x) = f(x)$$

II

If $f(x) = F'(x)$

$$\int_{[a,b]} f = F(b) - F(a)$$

A regular curve and its repar

$$\gamma_1 \circ \theta = \gamma_2 \quad \text{or} \quad \gamma_2 \circ \theta^{-1} = \gamma_1$$

then γ_2 is also regular.

$$\gamma_2' = (\gamma_1' \circ \theta) \times \theta'$$

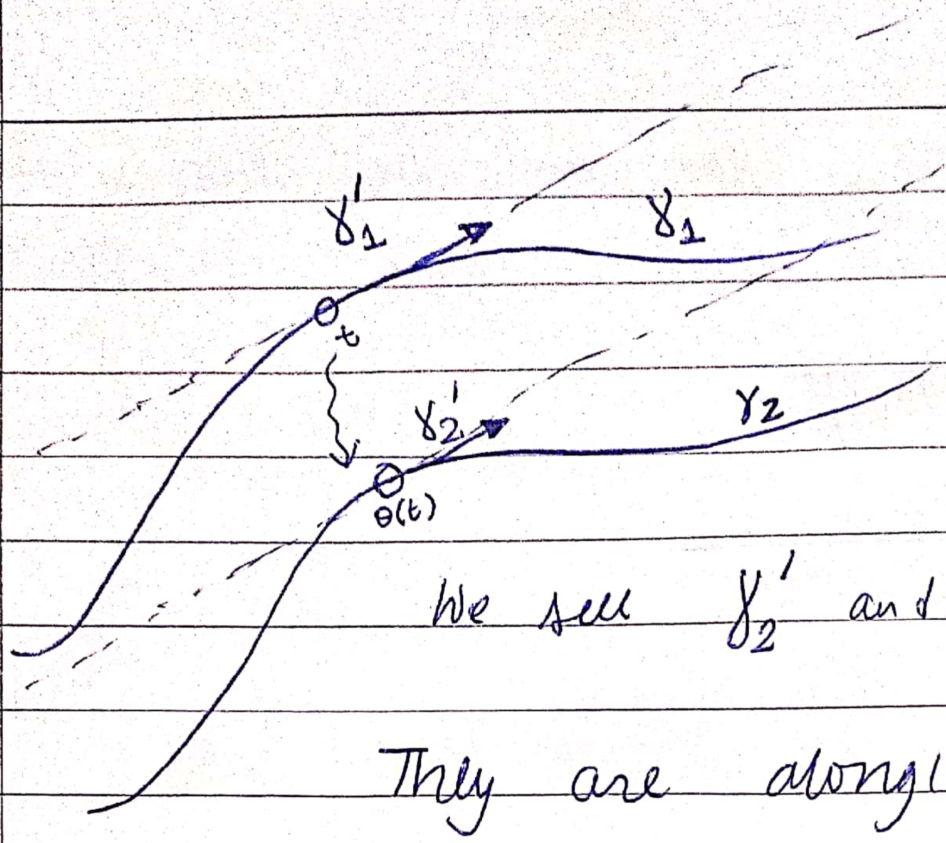
if

$$s = \theta(t) \Rightarrow t = \theta^{-1}(s)$$

$$\frac{ds}{dt} = \frac{d\theta}{dt} \qquad \frac{dt}{ds} = \frac{d\theta^{-1}(s)}{ds}$$

$$\frac{d(\theta^{-1})}{ds} = \frac{1}{\frac{d\theta}{dt}}$$

29AUG22



$\gamma_2 = \gamma_1 \circ \theta$

$\gamma_2' = \theta' \cdot \gamma_1' \circ \theta$

Scalar multp. of functions ?

We see γ_2' and γ_1' are in same direction.

They are along(?) the <u>tangent space</u> / <u>tangent line</u> of the curve

For unit speed parameterization, say $\theta(t)$ gives us such a param.,

$$\theta'(t) = \frac{1}{\|\gamma_1'(\theta(t))\|} \quad ; \quad s := \theta(t) \Rightarrow t = \theta^{-1}(s)$$

$$\Rightarrow \int_{t_0}^{t_1} \|\gamma_1'(\theta(t))\| \theta'(t)\, dt = \int_{t_0}^{t_1} dt$$

$$\Rightarrow \int_{s_0 = \theta^{-1}(t_0)}^{s_1 = \theta^{-1}(t_1)} \|\gamma_1'(s)\|\, ds = \int_{t_0}^{t_1} dt$$

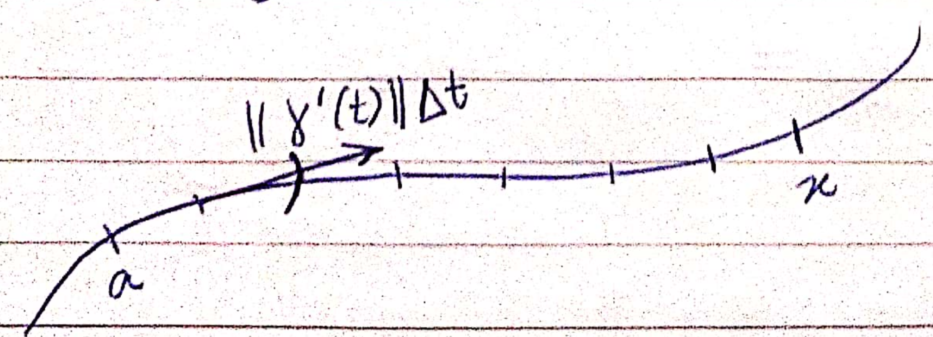
$$= t_1 - t_0$$

Let $F(x) := \int_a^x \|\gamma'(t)\|\, dt$

where γ is a reg-smooth param, $F(x)$ is the arc length fxn.

then $F(\theta^{-1}(t_1)) - F(\theta^{-1}(t_0)) = t_1 - t_0$

$\Rightarrow F \circ \theta^{-1} = \underline{Id}$



$F(x)$ is differentiable, from fundamental theorem of calculus, and $\gamma \in \mathcal{C}^\infty$

$$F'(x) = \|\gamma'(x)\|$$

$$= \sqrt{\gamma'(x) \cdot \gamma'(x)}$$

$F''(x)$ exists only if $\|\gamma'(x)\| \neq 0$, and

$$F''(x) = \frac{1/2}{\sqrt{\gamma'(x) \cdot \gamma'(x)}} \; 2\gamma''(x) \cdot \gamma'(x)$$

$$= \frac{\gamma''(x) \cdot \gamma'(x)}{\|\gamma'(x)\|}$$

$\Rightarrow$ $\boxed{F(x) \in C^\infty \quad \text{if} \quad \gamma \text{ is regular}}$

Exercise

Claim: θ that $\to F \circ \theta = \mathrm{Id}_I = \theta \circ F$

will be a unit speed reparam. (4)

proof $\to (\theta \circ F)(x) = x$

$\Rightarrow (\theta \circ F)'(x) = 1$

$\Rightarrow \theta'(F(x))\, F'(x) = 1$

$\Rightarrow \theta'(F(x)) = \dfrac{1}{F'(x)} = \dfrac{1}{\|\gamma'(x)\|}$

$F(x) = y \Rightarrow x = F^{-1}(y)$
$= \theta^{-1}(y)$

OR $\theta'(y) = \dfrac{1}{\|\gamma'(\theta(y))\|}$

Now, get such an $\theta(t)$, w/ $\theta' \|\gamma' \circ \theta\| = 1$

$$\gamma_2 := \gamma_1 \circ \theta \,,$$

$$\gamma_2' = (\gamma_1 \circ \theta)'$$

$$= \gamma_1' \circ \theta \cdot \theta'$$

$$\|\gamma_2'\| = \|\gamma_1' \circ \theta\| \, |\theta'|$$

$$= 1 .$$

Hence, we did it!

$\therefore \gamma_2 := \gamma_1 \circ F^{-1}$

a unit speed reparam!

31-10-22

$\int g(x)|f'(x)|dx \neq \int g\, dx$

$f(x) = u$

$|f| = \begin{cases} f & f > 0 \\ -f & f < 0 \end{cases}$

$\boxed{\text{sejdm.github.io/mth201}}$

$$S(x) := \int_a^x \|\gamma'(t)\|\, dt$$

$S(x) \longmapsto S \circ \theta^{-1}(x)$

$\boxed{\text{(Thm)}\quad S(x) \text{ does not depend on parameterization.}}$

proof

Say γ_1 and γ_2 are conn. by θ a CT.

~~Then~~ in : $\gamma_2 = \gamma_1 \circ \theta$

then $S[\gamma_1](x) = \int_{[a_0, x]} \|\gamma_1'\|$

$S[\gamma_2](s) = \int_{[s_0, s]} \|\gamma_2'\|$

$\neq 0 \neq$

$\boxed{\theta'(\theta^{-1}(t))}\ \boxed{\theta^{-1\prime}(t)}$ $= 1$

smooth $\begin{cases} \theta(t) \in \mathcal{C}^\infty \\ \theta^{-1} \in \mathcal{C}^\infty \end{cases}$

where $s_0 = \theta^{-1}(x_0)$

$s = \theta^{-1}(x)$.

$\frac{d}{dt}(\theta \circ \theta^{-1}) = \frac{d}{dt}(t) = 1$

Now, $\gamma_2' = \gamma_1' \circ \theta \cdot \theta'$ $\quad \theta'$ is either $+$ or $-$.

$\|\gamma_2\| = \|\gamma_1' \circ \theta\|\ |\theta'|$

θ'

$|\theta'| = \theta'$

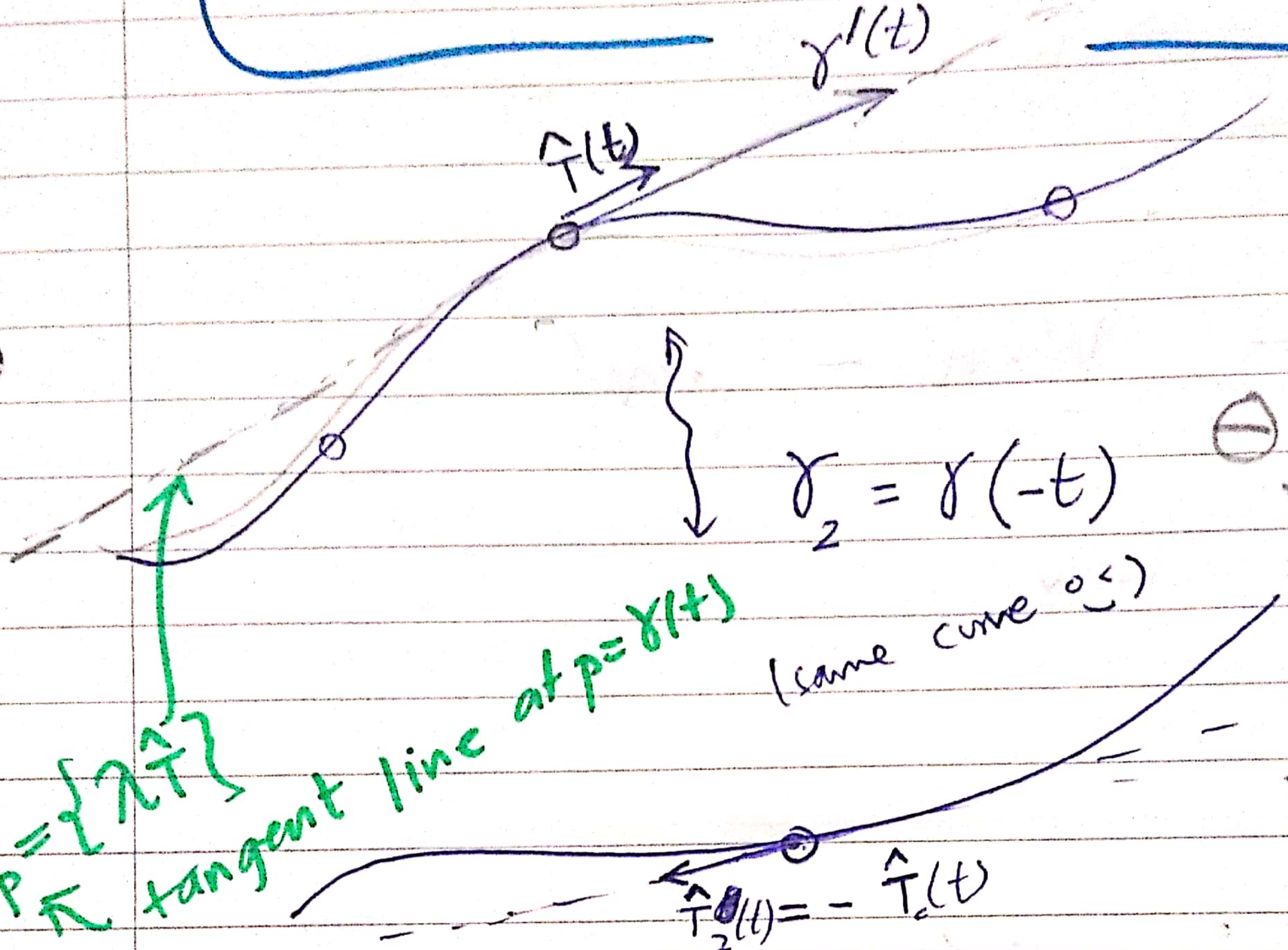
Hence, by composition of fn. integral

$$\boxed{S[\gamma_1](x) = S[\gamma_2](s)}.$$

6

(Def) $\gamma : [0, b] \to \mathbb{R}^3$

$$\hat{T}(t) := \frac{\gamma'(t)}{\|\gamma'(t)\|}$$ is defined to be the unit tangent vector of γ at t.



$\Theta(t) = -t$

$\gamma_2 := \gamma \circ \Theta$

$\hat{T}_2 = -\hat{T}$

$$\boxed{\text{Invariant under reparam?}}$$

$T_2 = T_1 \circ \Theta$

$\gamma_2 := \gamma_1 \circ \Theta$ be a CT.

then $\|\gamma_2'\| = \|\gamma_1' \circ \theta\| \, |\theta'|$

$$\frac{\gamma_2'}{\|\gamma_2'\|} = \frac{(\gamma_1' \circ \theta)\,\theta'}{\|\gamma_1' \circ \theta\| \, |\theta'|} = \frac{\pm \gamma_1' \circ \theta}{\|\gamma_1' \circ \theta\|}$$

$$\hat{T}_2(t) = \pm (\hat{T}_1 \circ \theta)(t)$$

~~assume all param are regular~~.

normal velocity

$$\boxed{\dot{\gamma}(t) = \underbrace{\|\dot{\gamma}(t)\|}_{} \hat{T}(t)}$$

$\frac{d}{dt}$ (arc length)

$$= S'(t)\, \hat{T}(t)$$

$$V : [a, b] \longrightarrow \mathbb{R}^3 , \qquad W : [c, d] \longrightarrow \mathbb{R}^3$$

$$V \cdot W(t) = \sum_i V_{x_i}(t)\, W_{x_i}(t)$$

$$(V \cdot W)' = \underline{V'} \cdot W + V \cdot W'$$

$$\cos\theta \Big|_{t_0} = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|\, \|\vec{w}\|}\Bigg|_{t_0}$$

$$V \cdot W = \underline{\|V\|}\, \underline{\|W\|} \cos\theta$$

$\|n(t)\| = \text{const.}$

$v(t)$ IF $\|\underline{v'(t)}\| = 1$

$\dot{n} \cdot n = 0$

$n(t) = \theta$

$$\Rightarrow V(t) \cdot V(t) = 1$$

$$\Rightarrow V'(t) \cdot V(t) + V(t) \cdot V'(t) = 0$$

$$\Rightarrow 2\, V'(t) \cdot V(t) = 0$$

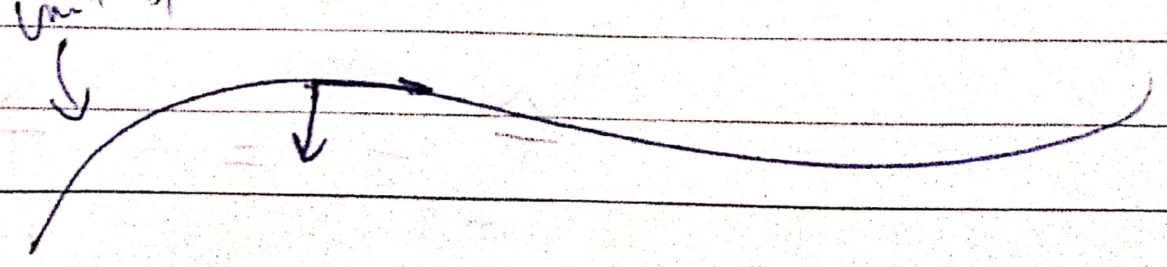
$$\Rightarrow \boxed{V'(t) \cdot V(t) = 0}$$

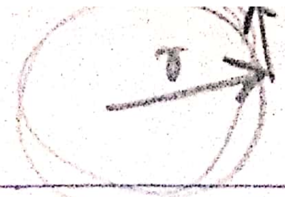
v'

v'

$\boxed{\text{orthogonal}}$

unit speed



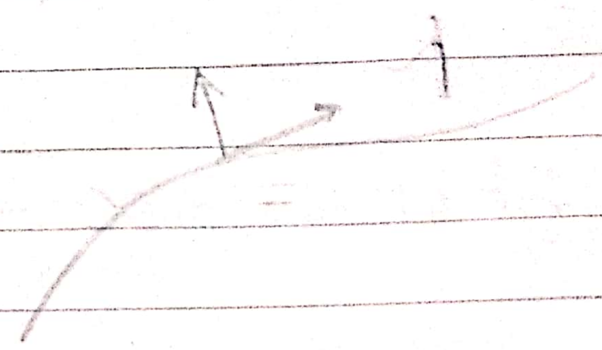


$\|\gamma(t)\| = R$

$\gamma \cdot \dot{\gamma}(t) = 0$

Hence, $\ddot{\gamma}(t)$ for unit speed param,

$$\ddot{\gamma} \cdot \dot{\gamma} = 0 .$$



2SEP22

* Thm: $(\gamma \circ s^{-1})$ is a unit speed reparam. of γ.

(Defn) Given a unit speed param. $\tilde{\gamma} := (\gamma \circ s^{-1})$

where $\tilde{\gamma} : (a, b) \to \mathbb{R}^3$

$\kappa(t) := \|\tilde{\gamma}''(t)\|$ called the curvature

at the point $\gamma(t)$.

$$s \circ f = x = f(s(x))$$
$$\Rightarrow f(Rx) = x$$
$$\Rightarrow f(y) = \frac{y}{R}$$

$$\gamma : (-1, 1) \longrightarrow \mathbb{R}$$
$$\gamma := (R\cos t,\ R\sin t)$$
$$\dot{\gamma} = (-R\sin t,\ R\cos t)$$
$$\|\dot{\gamma}\| = R$$
$$s := \int^{x} \|\dot{\gamma}\|\, dt = Rx - 0 = Rx$$

$$(-1, 1) \to (-R, R)$$

$$(-R, R) \longrightarrow (-1, 1).$$
$$s^{-1}(x) = \frac{x}{R}$$

$$\tilde{\gamma}(t) = (\gamma \circ s^{-1})(t) = \left(R\cos\left(\frac{t}{R}\right),\ R\sin\left(\frac{t}{R}\right) \right)$$

$\boxed{\tilde{\gamma} \text{ is unit speed param}}$.

$$(\tilde{\gamma}')(t) = \left(-\sin\left(\frac{t}{R}\right),\ \cos\left(\frac{t}{R}\right) \right)$$

$$(\tilde{\gamma}'')(t) = \left(-\frac{1}{R}\cos\left(\frac{t}{R}\right),\ -\frac{1}{R}\cos\left(\frac{t}{R}\right) \right)$$

$$\|\tilde{\gamma}''\|(t) = \frac{1}{R}$$

$\Theta_{US} := S^{-1}$ $\qquad \frac{d}{dt}\tilde{\gamma}_x(t) = \frac{d}{dt}(\|\gamma(t)\| \; \tilde{\gamma}_x(s(t)))$

$$\tilde{\gamma} := \gamma \circ S^{-1}$$

$$\Rightarrow \tilde{\gamma} \circ S = \gamma$$

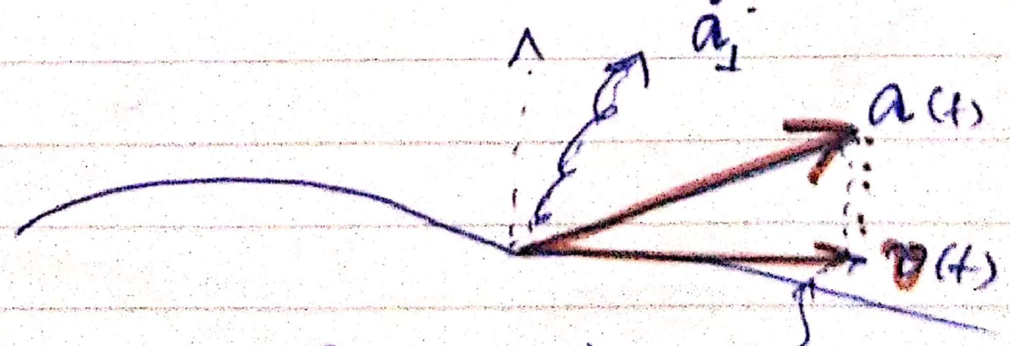
$$\Rightarrow \gamma' = (\tilde{\gamma} \circ s)' = (\tilde{\gamma}' \circ s) \cdot \underbrace{(s')}_{\|\gamma'\|}$$

$$\Rightarrow \gamma' = \|\gamma'\| (\tilde{\gamma}' \circ s)$$

$$\Rightarrow \gamma'' = \underbrace{\frac{d}{dt}(\|\gamma'(t)\|)}_{\frac{\gamma''(t) \cdot \gamma'(t)}{\|\gamma'(t)\|}} (\tilde{\gamma}' \circ s) + \|\gamma'\| \underbrace{(\tilde{\gamma}' \circ s)'}_{\underbrace{s'}_{\|\gamma'\|} \cdot (\tilde{\gamma}'' \circ s)}$$

$$\therefore \gamma''(t) = (\tilde{\gamma}' \circ s) \gamma'' \cdot \frac{\gamma'}{\|\gamma'\|} + \|\gamma'\|^2 \underbrace{(\tilde{\gamma}'' \circ s)}$$

$$\Rightarrow \|\gamma'\|^2 (\tilde{\gamma}'' \circ s)(t) = \underbrace{\gamma''(t)}_{\vec{a}_\perp ?} - \gamma'' \cdot \underbrace{\frac{\gamma'}{\|\gamma'\|}(t)}_{\hat{T}(t)} \underbrace{(\tilde{\gamma}' \circ s)}_{T(t)}$$



k =

$$\vec{a}_\perp = \vec{a} - a_{\parallel}(t) \hat{T}$$

Now $\|\gamma'' \circ s\| = \kappa \circ s$

$$\text{"}\frac{1}{R}\text{"} = \frac{a}{v^2} \qquad \frac{v^2}{R} = \|a\|$$

$a_\perp$

$$\gamma''\circ s = \frac{\gamma'' - (\gamma''\cdot\hat{T})\hat{T}}{\|\gamma'\|^2}$$

$$K = \tilde{K}\circ s = \|\gamma''\circ s\|$$

$$= \frac{\|\gamma'' - (\gamma''\cdot\hat{T})\hat{T}\|}{\|\gamma'\|^2} \leftarrow \frac{\|a_\perp\|}{\|v\|^2}$$

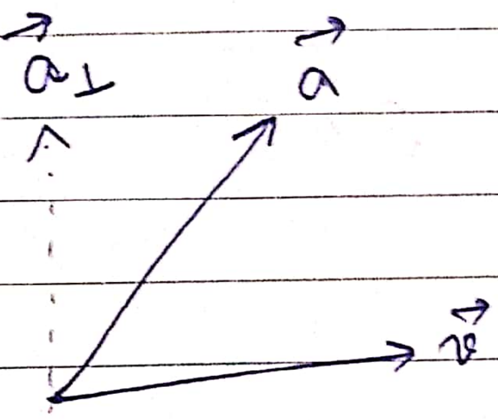
$$\tilde{\gamma}'\circ s$$

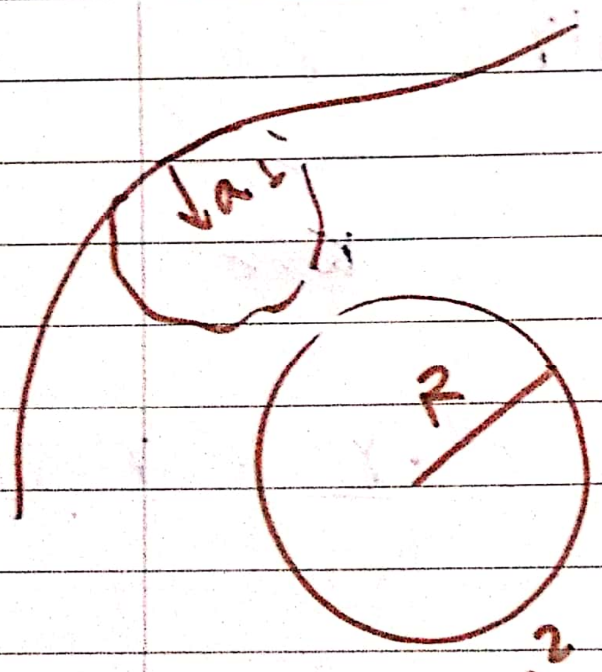
$$\tilde{\gamma} = \gamma\circ s^{-1}$$

$$\tilde{\gamma}\circ s = \gamma$$

$$s'\,(\tilde{\gamma}'\circ s) = \gamma'$$

$$\tilde{\gamma}'\circ s = \frac{\gamma'}{\|\gamma'\|}$$





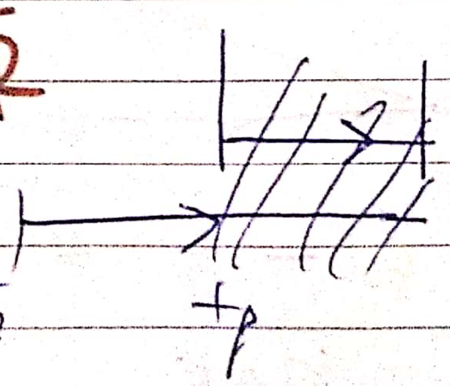
$$\frac{\|a_\perp\|^2}{\|v\|^2}$$

$$\boxed{K = \frac{\|a_\perp\|}{\|v\|^2}}$$

$$\int_a^b f(t) \neq \int_b^d f(t)$$

$$\frac{v^2}{R} = a$$

$$\text{"}\frac{1}{R}\text{"} = \frac{a}{v^2}$$



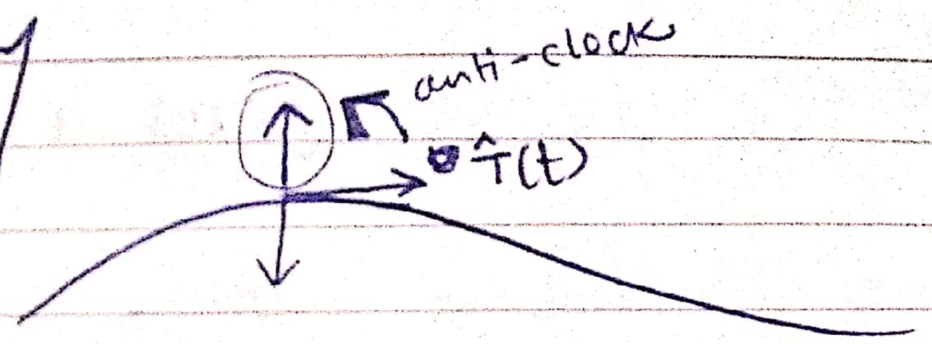
$$K := \cancel{\ldots}\,\|\tilde{\gamma}''\|$$

$$\hat{N} := \frac{\tilde{\gamma}''}{\|\tilde{\gamma}''\|}$$

$\gamma : (a,b) \to \mathbb{R}^2$

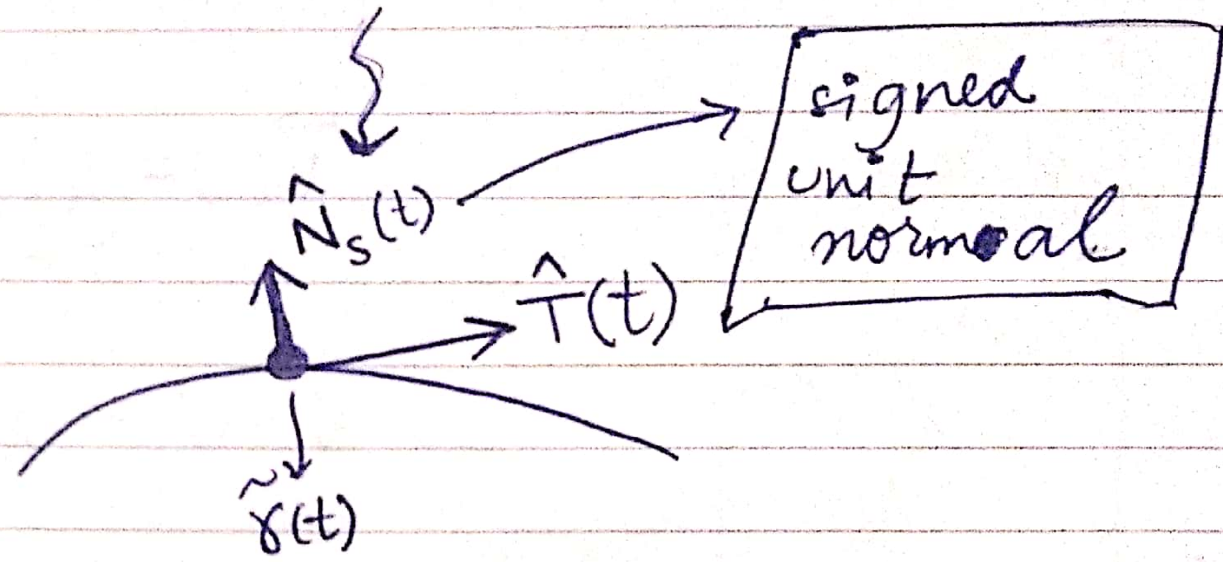
5SEP22

PLANE curves



$\{\hat{T}, \hat{N}\}$

$\{\hat{T}, \hat{N}_s\}$



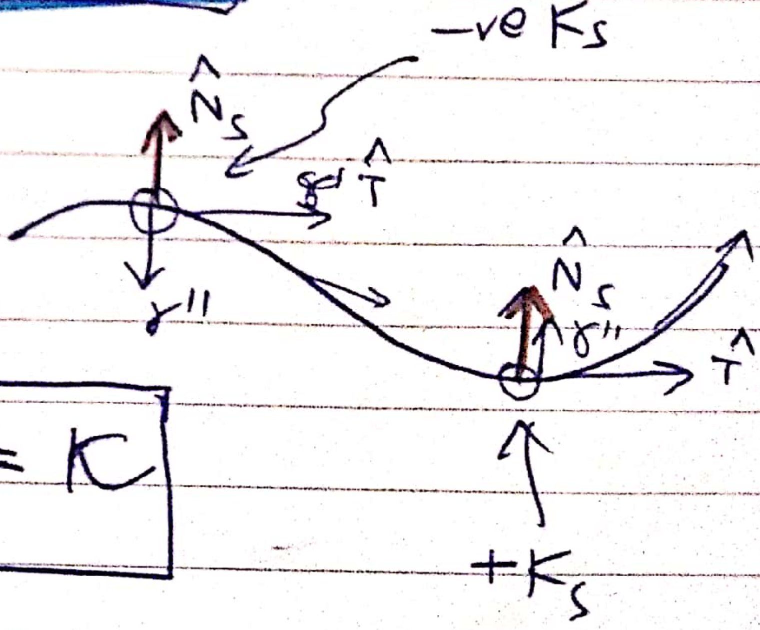
Let $\tilde{\gamma}$ be unit speed,

then $\hat{T}(t) = \tilde{\gamma}'(t)$.

and $\tilde{\gamma}''(t) \cdot \hat{T}(t) = 0$

Now define $\boxed{\tilde{\gamma}''(t) := \kappa_s(t)\, \hat{N}_s(t)}$

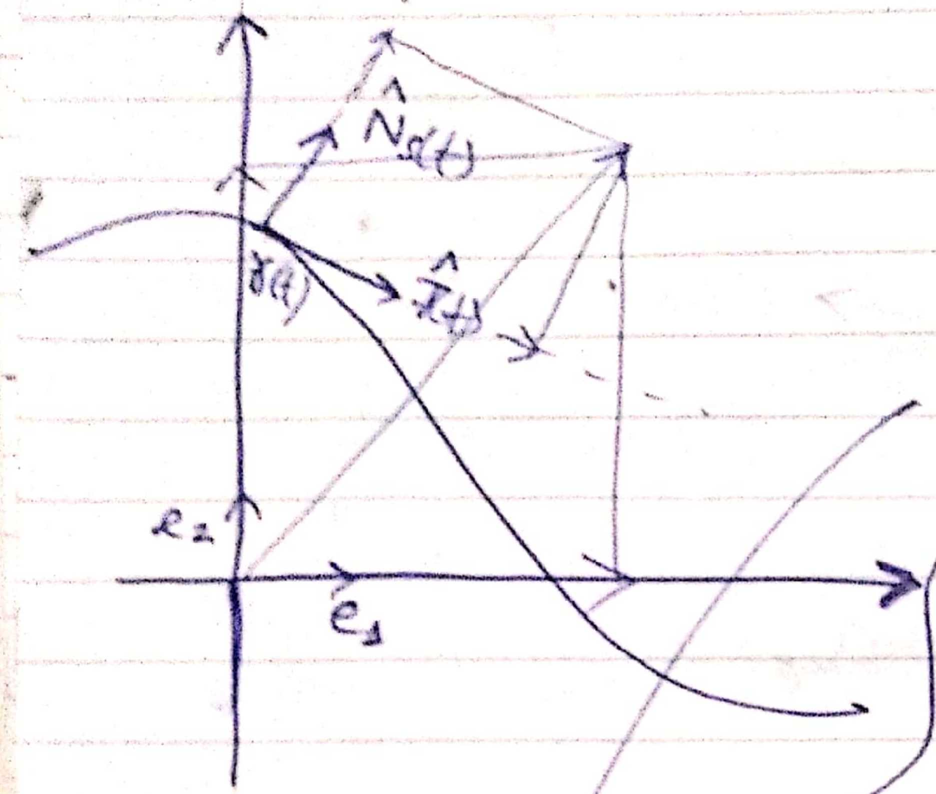
Signed curvature.



$\boxed{\text{Notice: } |\kappa_s| = \kappa}$

$\boxed{\hat{T}'(t) = \kappa_s(t)\, \hat{N}_s(t)}$

* $\hat{T}(t)$, $N_s(t)$ form a orthonormal basis for Euclidean $\mathbb{R}^2$, "based at $\gamma(t)$".



$$\vec{v}(t) = \underbrace{x(t)}\,\hat{T}(t) + \underbrace{y(t)}\,\hat{N}_s(t)$$

$$\vec{v}'(t) = x'(t)\,\hat{T}(t) + x(t)\,\underbrace{\hat{T}'(t)}$$
$$+\ y'(t)\,\hat{N}_s(t) + y(t)\,\hat{N}_s'(t).$$

$$\hat{T}(t) = \gamma'(t)$$
$$\hat{T}'(t) = \gamma''(t) = K_s(t)\,\hat{N}_s(t)$$
$$N_s'(t) = \underbrace{(\ ?\ )}\,\hat{T}(t).$$

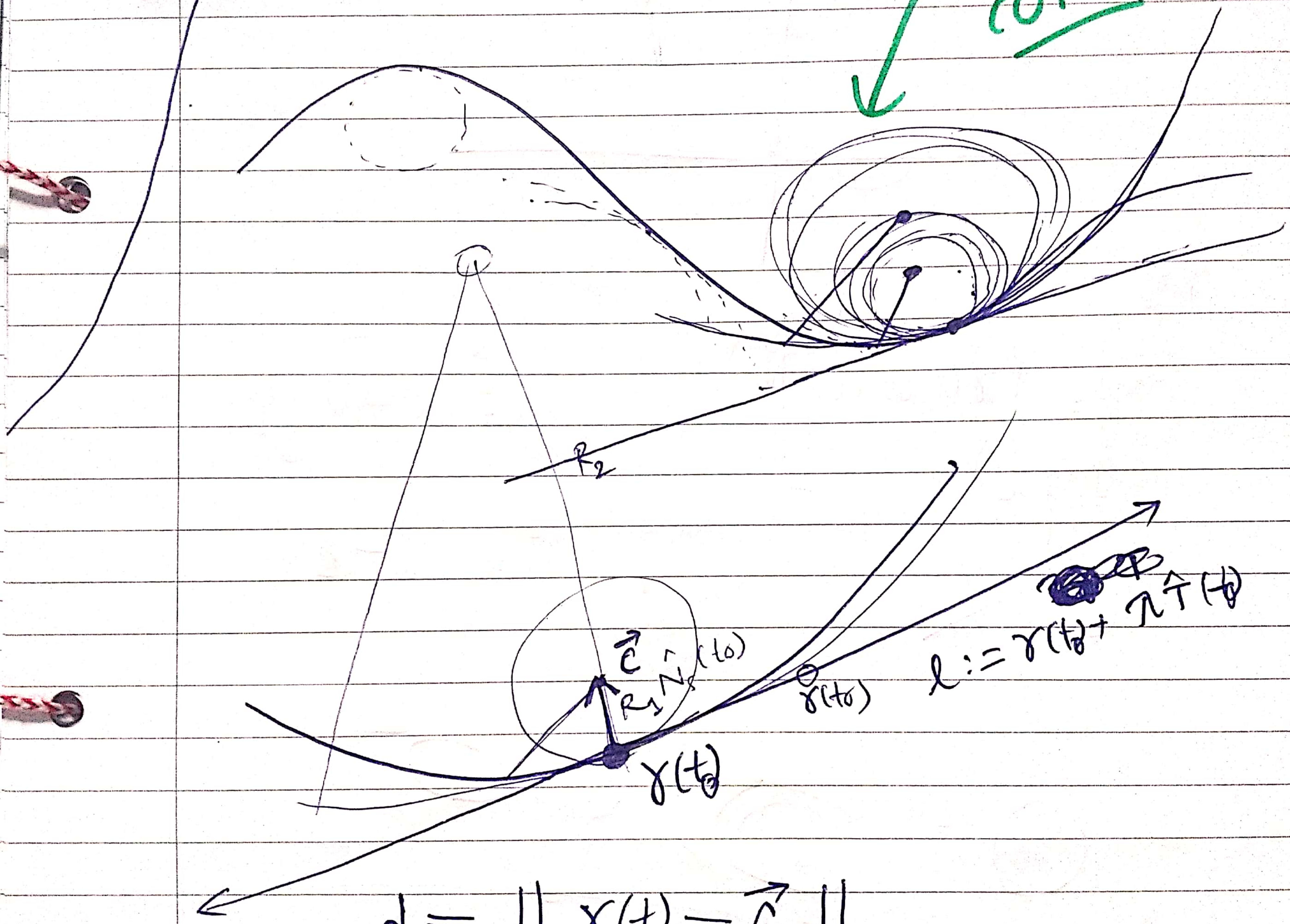
$$N_s'(t)\cdot\hat{T}(t)$$
$$= -N_s(t)\cdot T_s'(t)$$
$$= -K_s(t).$$

$$\vec{v}'(t) = \cancel{\ \ }$$
$$\big(x'(t) + y(t)(-K_s(t))\big)\,\hat{T}(t)$$
$$+\ \big(y'(t) + x(t)\cdot K_s(t)\big)\,\hat{N}_s(t).$$

$$\underbrace{(N_s(t)\cdot\hat{T}_s(t))'}_{0} = N_s'(t)\cdot T_s(t) + N_s(t)\cdot T_s'(t)$$

$$\boxed{\begin{aligned}\hat{T}'(t) &= K_s(t)\hat{N}_s(t)\\ \hat{N}_s'(t) &= -K_s(t)\,\hat{T}(t)\end{aligned}}$$

best circle to approximate the curve



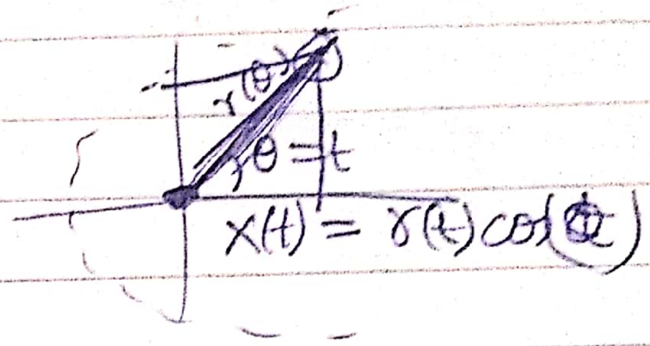
$$d_c = \|\gamma(t) - \vec{C}_R\|$$

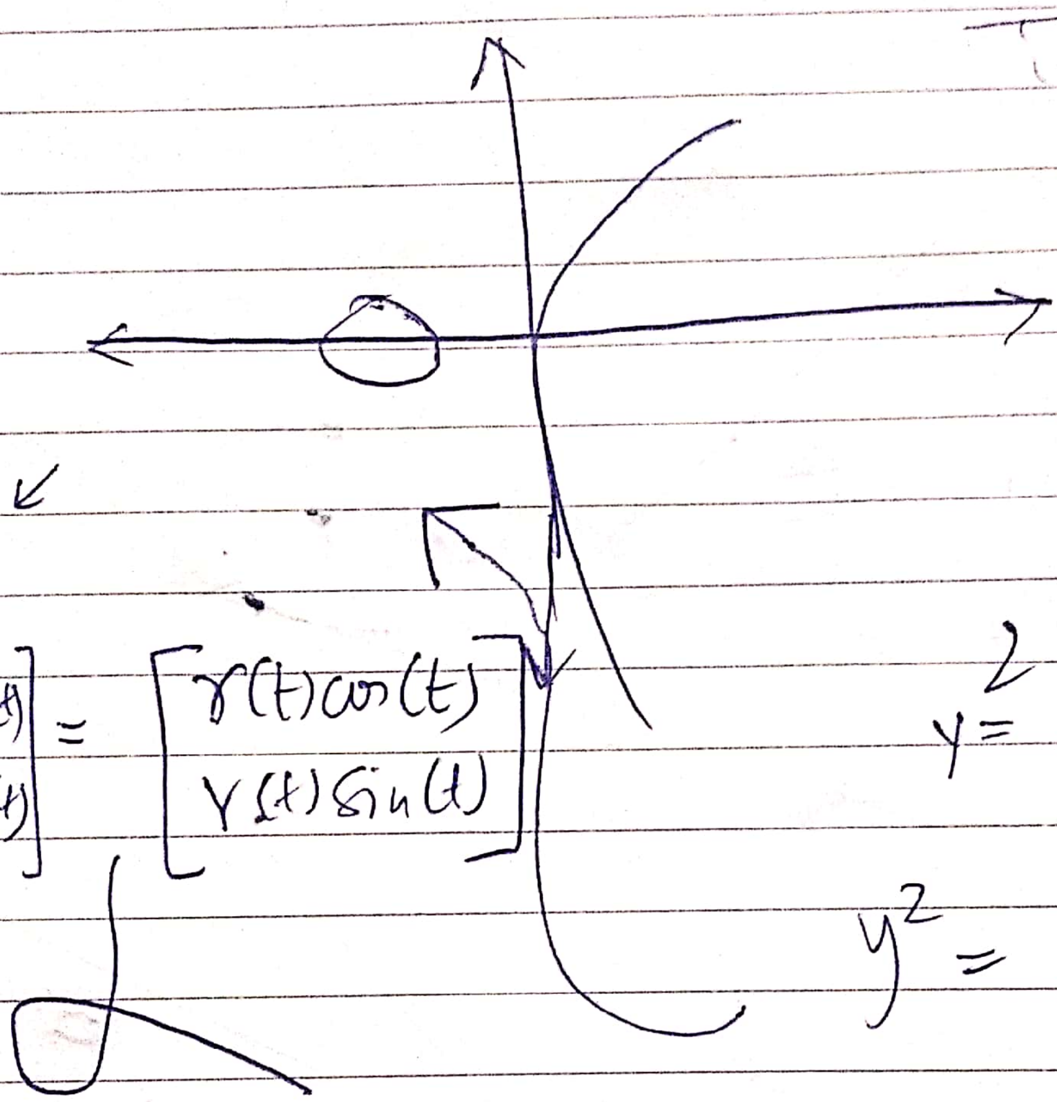
$$\gamma(t_0) + R\hat{N}_s(t_0)$$

$$d(t) = \|\gamma(t) - (\gamma(t_0) - R\hat{N}_s(t_0))\|$$

$$\frac{d}{d\theta}\left(d_R^2(\theta)\right) = 0 \quad \text{and} \quad \frac{d^2}{d\theta^2}\left(d_R^2\right) =$$

$\Longrightarrow$ R will be $\frac{1}{\kappa}$!



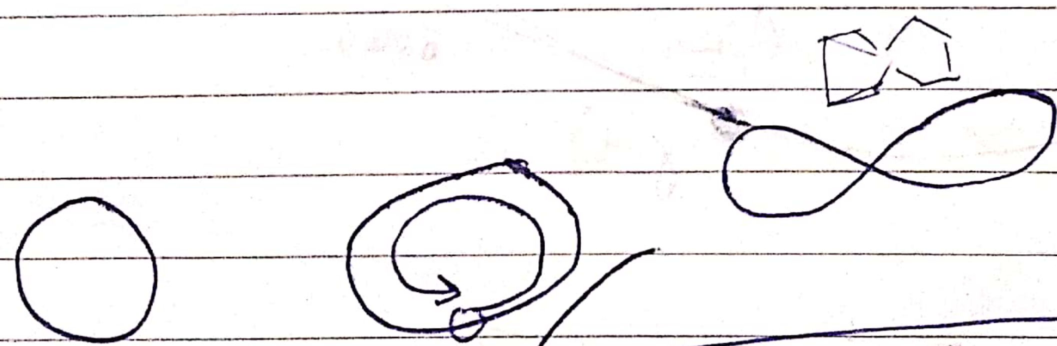


$r(\theta)$

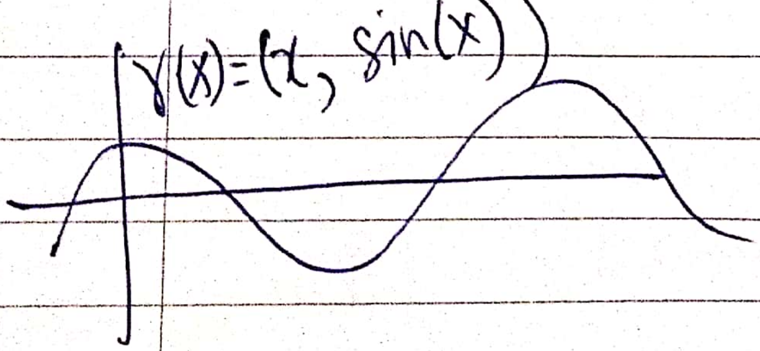
$$\gamma(t)=\begin{bmatrix}x(t)\\ y(t)\end{bmatrix}=\begin{bmatrix}r(t)\cos(t)\\ r(t)\sin(t)\end{bmatrix}$$

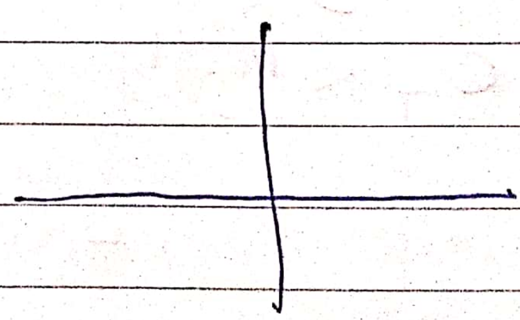
$$y^2 = x^3 + ax + b$$

$$y^2 = x^3 - x$$



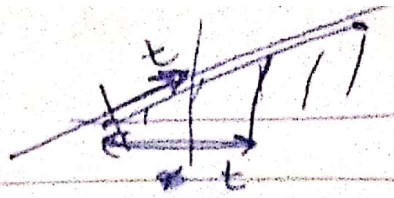
$$\boxed{\gamma(t) = \gamma(t+T)}$$





MTH201 Ex 1

$\gamma(t) = \gamma(0) + \lambda v$



$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ y(t) \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix}(t)$

1. $\gamma(t) := (x_1, y_1)\, t + (1-t)\,(x_2, y_2)$

Here, $\gamma(0) = (x_2, y_2)$
$\gamma(1) = (x_1, y_1)$

$\gamma : [0, 1] \longrightarrow \mathbb{R}^2$ represents the line segment joining the two points.

$\dot\gamma(t) = (x_1, y_1) - (x_2, y_2)$

2. $\gamma(t) := \left(\underbrace{\frac{2t}{1+t^2}}_{:= x(t)}, \underbrace{\frac{1-t^2}{1+t^2}}_{:= y(t)} \right)$

$$x^2 + y^2 = \frac{4t^2 + t^4 - 2t^2 + 1}{(1+t^2)^2}$$

$$= \frac{t^4 + 2t^2 + 1}{(1+t^2)^2} = 1.$$

~~A circle~~

A circle !

3. $\gamma(a) = \begin{bmatrix} a^2 - 1 \\ a(a^2-1) \end{bmatrix}$; If $a(a^2-1) = -a(a^2-1)$

$\gamma(-a) = \begin{bmatrix} a^2 - 1 \\ -a(a^2-1) \end{bmatrix}$ then $a^2 - 1 = 0$

makes it possible
$a = \pm 1$

$\therefore \gamma(1) = \gamma(-1)$. Hence, γ is not injective.

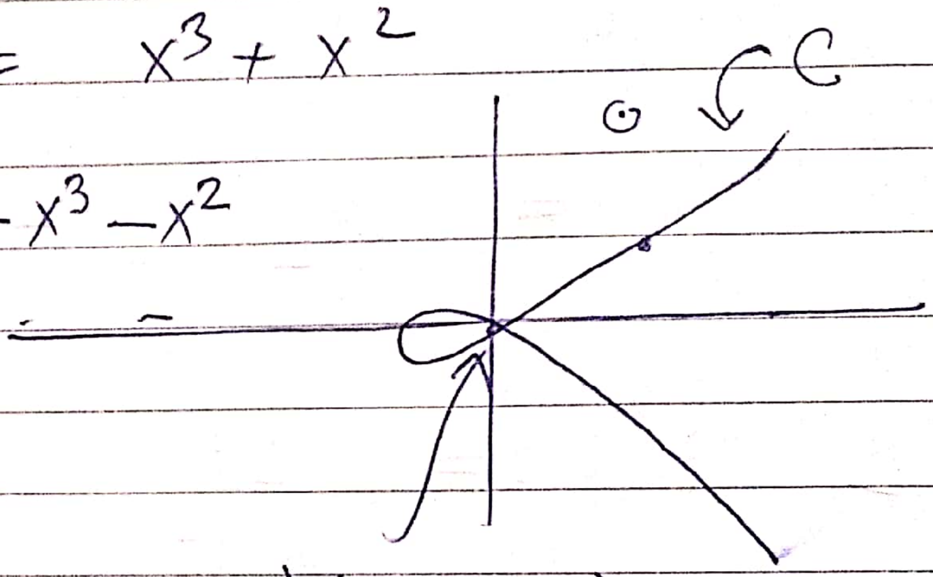
$(\sin t, \cos t)$
$=x \quad\quad =y \quad\quad x^2+y^2=1$

~~4. $f(t) := \vec{V}(t)\cdot\vec{W}(t)$~~

$x := t^2 - 1 \Rightarrow t = \sqrt{x+1}$
$y := t(t^2-1)$

$$y^2 = x^2(x+1)$$

$$f \quad\quad y^2 = x^3 + x^2$$



Hence $f\begin{bmatrix} x \\ y \end{bmatrix} := y^2 - x^3 - x^2$

$f: \mathbb{R}^2 \to \mathbb{R}$

$C := \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : f\begin{bmatrix} x \\ y \end{bmatrix} = 0 \right\}$

$\gamma(1) = \gamma(-1)$

Let

4. $f(t) := V(t)\cdot W(t) = \sum_i V_i(t) W_i(t)$

$\cancel{f: \mathbb{R} \to \mathbb{R}}$ $f: (\alpha,\beta) \to \mathbb{R}^2$

$$\frac{df}{dt} = \frac{d}{dt}\left(\sum V_i(t) W_i(t)\right)$$

$$= \sum_i \left[\left(\frac{d}{dt} v_i(t)\right) W_i(t) + v_i W_i'(t) \right]$$

$$= V'\cdot W + V\cdot W'$$

$$t_1^2 - 1 = t_2^2 - 1 \qquad \gamma(a) \quad \gamma(-a) \to \begin{bmatrix} a^2-1 \\ -a(a^2-1) \end{bmatrix}$$

$$t_1^2 = t_2^2 = a^2$$

$$a = \pm a$$

$$-a\,\cancel{(a^2-1)} = +a\,\cancel{(a^2-1)}$$

$$\frac{2t}{1+t^2} = x(t) \qquad -a = +a$$

$$a = 0 \qquad \underline{\gamma(0) = \gamma(0)}$$

$$\frac{1-t^2}{1+t^2} = y(t)$$

$$a^2 - 1 = 0$$

$$\Rightarrow a = \pm 1$$

$$\cancel{\frac{x}{y} = \frac{2t}{1+t^2}} \qquad \gamma(1), \quad \gamma(-1) \quad \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : \underline{f\begin{bmatrix} x \\ y \end{bmatrix} = 0} \right\}$$

$$(1-t^2)^2 = 1 - 2t^2 + t^4 \qquad f\begin{bmatrix} x \\ y \end{bmatrix} = c$$

$$+4t^2 \qquad\qquad +4t^2$$

$$= 1 + 2t^2 + t^4$$

$$= (t^2+1)^2 \qquad f\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$x^2 + y^2 = \frac{(2t)^2 + (1-t^2)^2}{(1+t^2)^2} = 1$$

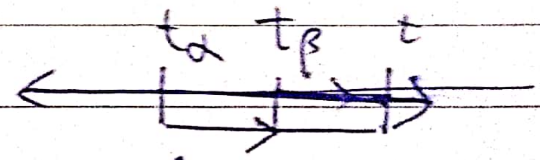
$$\|n\| = c$$

5. $$\Rightarrow n \cdot n = c^2$$

$$\Rightarrow n' \cdot n + n \cdot n' = 0$$

$$\Rightarrow n' \cdot n = 0.$$

$$\Rightarrow n' \perp n \text{ or } n' = 0$$



6. $$s_\beta(t) - s_\alpha(t) = \int_{t_\beta}^{t} \|\dot\gamma\| - \int_{t_\alpha}^{t} \|\dot\gamma\|$$

$$= \int_{t_\alpha}^{t_\beta} \|\dot\gamma\|$$

$$= \text{const.}$$

SPACE CURVES $\leftarrow \gamma : (a,b) \to \mathbb{R}^3$

$\tilde{\gamma}$ be a unit-speed param. $(\theta := \pm t + t_0)$

* $\hat{T}(t) := \tilde{\gamma}'(t)$; $\kappa(t) := \|\tilde{\gamma}''\|$

basis!

$\hat{T} \mapsto \pm \hat{T} \circ \theta$

* $\hat{N}(t) := \dfrac{\tilde{\gamma}''(t)}{\underbrace{\|\tilde{\gamma}''\|}_{\kappa}} \in \hat{T}'(t)$ $\leftarrow$ restrict to $\kappa \neq 0$ curves.

$\Rightarrow$ $\boxed{\hat{T}'(t) = \kappa(t)\,\hat{N}(t)}$

* $\hat{B}(t) := \hat{T}(t) \times \hat{N}(t)$

* $\hat{N}'(t) = -\kappa(t)\hat{T}(t) + \tau(t)\,\hat{B}(t)$

* $\hat{B}'(t) = \cancel{0\,\hat{T}(t)} + (-\tau(t))\,\hat{N}(t)$

$N \cdot T = 0$

$\Rightarrow N' \cdot \hat{T} + N \cdot \hat{T}' = 0$

$\Rightarrow N' \cdot \hat{T} = -\underbrace{N \cdot \hat{T}'}_{\kappa}$

$N \cdot B = 0$

$\Rightarrow \hat{N}' \cdot \hat{B} = -B' \cdot \hat{N}$

$B \cdot T = 0$

$\Rightarrow \underbrace{B' \cdot T} + \underbrace{T' \cdot B} = 0$

$\Rightarrow B' \cdot T = -0$

$= 0.$

$$\begin{bmatrix} \hat{T}' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \hat{T} \\ \hat{N} \\ \hat{B} \end{bmatrix}$$

* $|\tau| = \|B'\|$

(Thm) $\tau(t) = 0$ $\forall t$. $\Leftrightarrow$ $\tilde{\gamma}(t)$ is a plane curve.

Given a point γ_0 and a normal $\hat{n}$, the plane is

$$\text{Plane}(\hat{n}, \gamma_0) := \{\gamma : \hat{n}\cdot(\gamma - \gamma_0) = 0\}$$

Plane in 3D

proof

? Claim: $\hat{B}(t_0)\cdot\left(\tilde{\gamma}(t) - \tilde{\gamma}(t_0)\right) = 0$

Assume $\tau(t) = 0 \ \forall t$,

$\Rightarrow B'(t) = 0$.

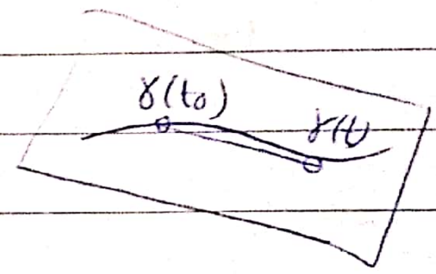
$\Rightarrow B(t) = B(t_0), \ \forall t$.

$\cancel{\gamma(t_0)\cdot B(t)}$

Vary: $\hat{T}\cdot B$

$\cancel{\gamma'\cdot B(t) = 0}$

$\cancel{\Rightarrow (\gamma(t) - \gamma(t_0))\cdot B}$



$$\frac{d}{dt}(\gamma(t)\cdot N(t)) = \gamma'(t)\cdot N + \gamma\cdot \underbrace{N'}_{(-\kappa \hat{T} \oplus)} = 0$$

$$\frac{d}{dt}\left(\gamma(t) - \gamma(t_0)\right)\cdot B(t_0) = 0$$

$$\Rightarrow \left(\gamma(t) - \gamma(t_0)\right)\cdot B(t_0) = \text{const}.$$

$$= \left(\gamma(t_0) - \gamma(t_0)\right)\cdot B(t_0)$$

$$= 0.$$

$\Rightarrow$ Hence $\exists\, n$ s.t.

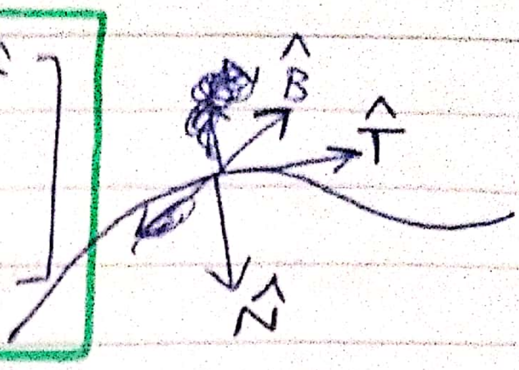
$$\hat{n}\cdot[\gamma(t) - \gamma(t_0)] = 0.$$

12 SEP 22

Frenet–Serret Equations

defn
$$\hat T := \gamma' \qquad \hat N := \frac{\gamma''}{\|\gamma''\|} \qquad \hat B := \hat T \times \hat N$$

$$\boxed{\begin{bmatrix}\hat T' \\ \hat N' \\ \hat B'\end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & +\tau \\ 0 & -\tau & 0\end{bmatrix}\begin{bmatrix}\hat T \\ \hat N \\ \hat B\end{bmatrix}}$$



(Thm) $\boxed{\kappa = 0} \iff \|\gamma''\| = 0$
$\iff \gamma'' = \vec 0 \in \mathbb{R}^3$
$\iff \gamma' = \text{const. } (\vec V_0)$; $\|\vec V_0\| = 1$
$\iff \boxed{\gamma = \vec V_0 t + \gamma(0)} \to$ line

(plane curves)

(Thm) $\boxed{\kappa_s = \text{constant} \iff \text{Curve lies on a circle.}}$

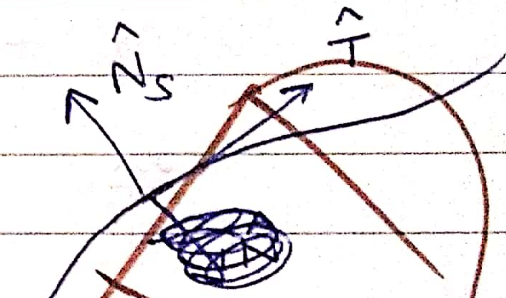
~~proof 1~~

$\kappa_s = c \iff$ ~~$\|\gamma''\| = c$~~

~~$c \neq 0$~~

~~$\iff$~~

$c \neq 0$



$p(t) := \gamma - R N_s$

$p' = \underbrace{\gamma'}_{\hat T} - R \underbrace{N_s'}_{\kappa_s \hat T}$

$\gamma' = (1 - R\kappa_s)\, \hat T$

Keep $R\kappa := \frac{1}{|\kappa_s|}$

then $p' = 0$

$\Rightarrow p(t) = p_0$

$\|\gamma - p_0\| = \frac{1}{\cdot}$ (const) $\to$ Hence it is a circle

~~$q(t) := \|\gamma - R\hat N_s\|^2$~~

~~$q' = 2(\gamma - R\hat N_s)\cdot(\gamma' - R\kappa_s \hat T)$~~

~~$= 2\big(\gamma\cdot\gamma' - \gamma\cdot\hat T R\kappa_s - R\hat N_s\cdot\gamma' + R^2\kappa_s\big)$~~

~~$= 2\big(\gamma\cdot\gamma' - \gamma\cdot\gamma' R\kappa_s\big)$~~

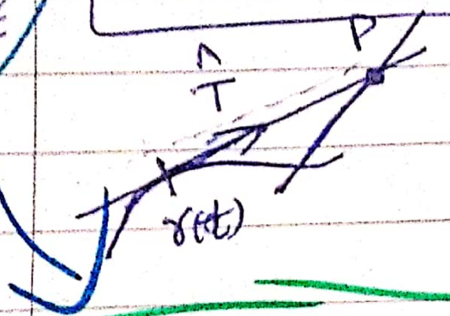
~~put $R := \frac{1}{|c|}$~~

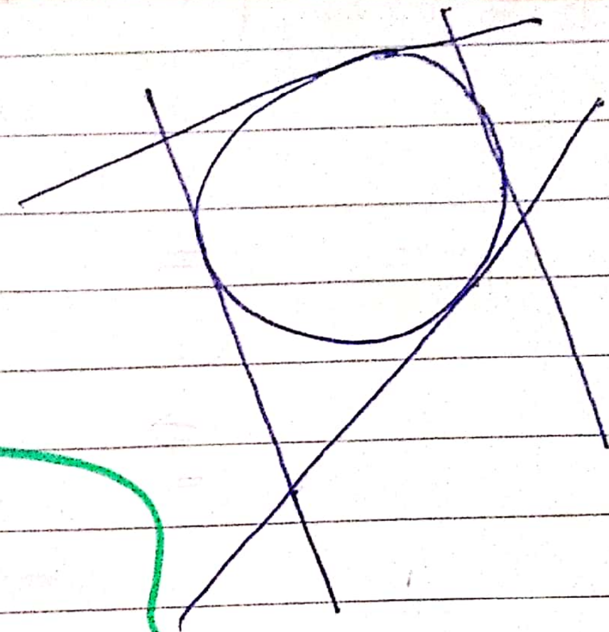
~~$q' = 0$~~

Space

(Thm) IF all the tangent lines to a curve pass through a fixed point p, then the curve must lie on a line.

(Proof) $\tilde{\gamma}$ be USP





Tangent line $\hookrightarrow$ $\gamma(t) + \lambda \hat{T}(t)$

at t for some λ_p, $\underline{\gamma(t) + \underbrace{\lambda_p}_{\lambda_p(t)} \hat{T}(t) = p}$.

$$\underbrace{\gamma'}_{\hat{T}} + \lambda_p' \hat{T} + \lambda_p \underbrace{\hat{T}'}_{+K\hat{N}} = \vec{0}$$

$$\Rightarrow (1+\lambda_p') \hat{T}(t) + K\lambda_p \hat{N}(t) = \vec{0}.$$

$\Rightarrow$

$\cancel{1+\lambda_p' = 0}$; $K\lambda_p = 0$

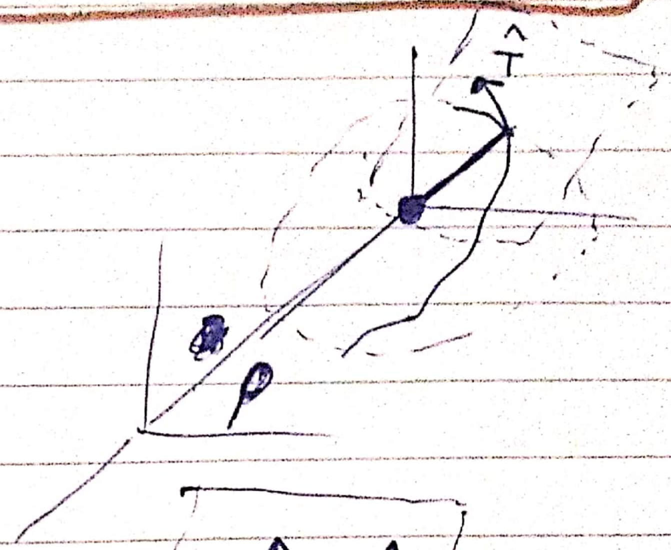
$\Rightarrow$ $\cancel{\lambda_p' = -1}$ $\Rightarrow$ either $K = 0$

or $\lambda_p = 0$ $\leftarrow$ not possible,

$\Rightarrow$ $K(t) = 0$ $\Rightarrow$ lies on a $\underline{\text{line}}$.

space

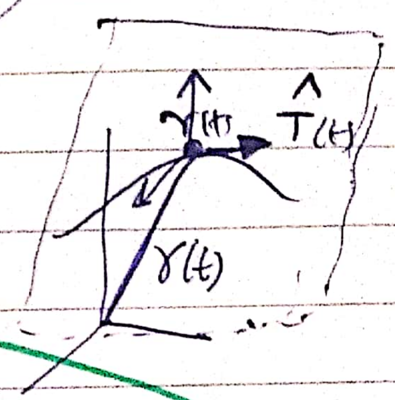
$\boxed{\text{(Thm) If Every plane normal to } \hat{T}(t) \text{ of a given curve contains a point } \underline{p}, \text{ then curve lies on a sphere.}}$



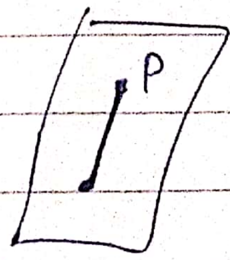
proof

$\rightarrow$ $\tilde{\gamma}$ be USP.

$\cancel{\tilde{\gamma} \cdot \tilde{\gamma}' = \frac{1}{2}\frac{d}{dt}\|\tilde{\gamma}\|^2 = \frac{1}{2}\frac{d}{dt}\tilde{\gamma}\cdot\tilde{\gamma}}$



$P_x^{(t)} := (\gamma(t) - x) \cdot \hat{T}(t)$



$p \in \{P_x^{(t)}\} \quad \forall t.$

$\Rightarrow \underline{(\tilde{\gamma}(t) - p) \cdot \underbrace{\hat{T}}_{\tilde{\gamma}'} = 0} \quad \forall t.$

$\cancel{\Rightarrow \tilde{\gamma} \cdot \tilde{\gamma}' = p \cdot \hat{T}}$

$(\tilde{\gamma} - p)' = \tilde{\gamma}'$

$\Rightarrow (\tilde{\gamma} - p) \cdot \tilde{\gamma}' = 0$

$\cancel{\frac{d}{dt}\left[(\gamma - p) \cdot \gamma\right] =}$

$\cancel{= (\gamma' \cdot \gamma) + \underbrace{(\gamma - p) \cdot \gamma'}_{0}}$

$(\tilde{\gamma} - p) \cdot (\tilde{\gamma} - p)' = 0$

$\Rightarrow \|\tilde{\gamma} - p\| = \text{const.}$

$(1,2) \cup (2,3)$

N° =

Quiz I

Rupadarshi Ray

MS21165

MTH201
Quiz 1

5/5 A.R.

Q Consider $\gamma : (0, \pi/2) \longrightarrow \mathbb{R}^2$ defined by

$$\gamma(t) := (3\cos t,\ 2\sin t)$$

1. The curve has coordinates $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3\cos t \\ 2\sin t \end{bmatrix}$

which follows

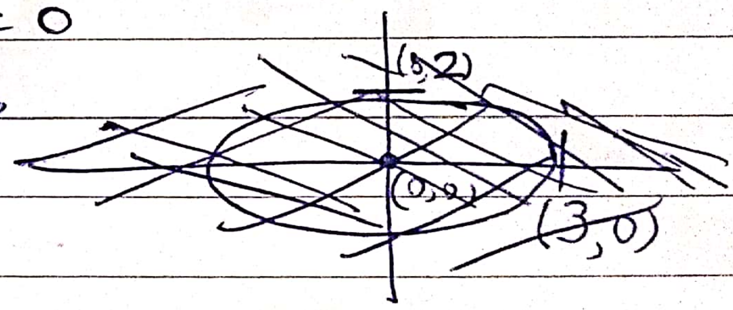
$$\frac{x^2}{3^2} + \frac{y^2}{2^2} = \cos^2 t + \sin^2 t = 1^2$$

Hence, $f\begin{pmatrix} x \\ y \end{pmatrix} := \frac{x^2}{3^2} + \frac{y^2}{2^2} - 1$ ✓ 1

~~the curve is $f\begin{pmatrix} x \\ y \end{pmatrix} = 0$, which is a circle of radius 3 and centered at $(0,0)$.~~

the curve is $f\begin{pmatrix} x \\ y \end{pmatrix} = 0$

which is an ellipse centered at $(0,0)$



where

The major axis (in x-axis) is of length 6 units

and minor axis (y-axis) is of length 4 units.

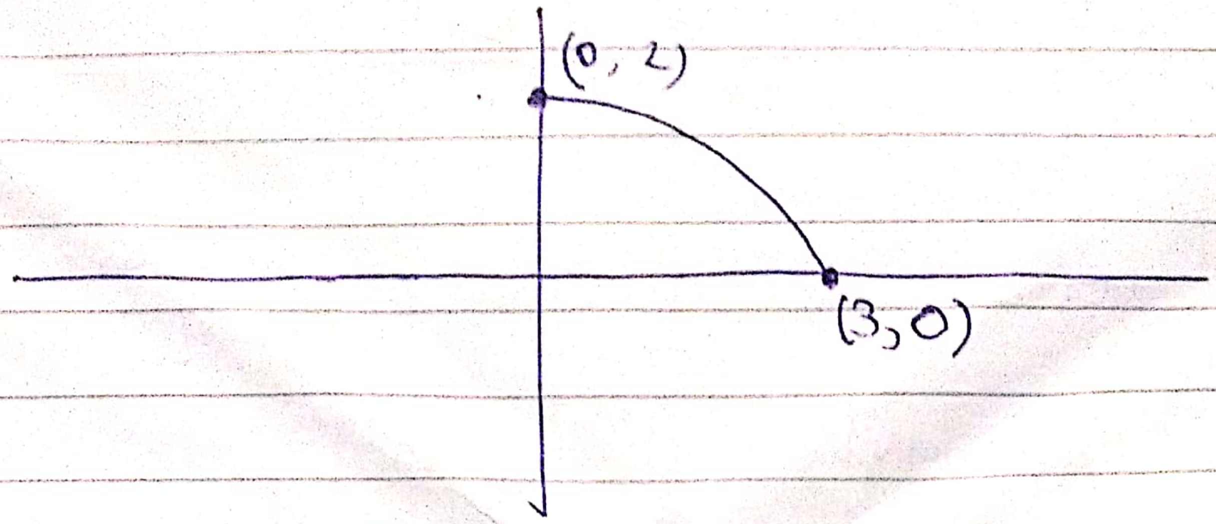
2. $\gamma(t) = (3\cos t,\ 2\sin t)$

$\gamma'(t) = (-3\sin t,\ 2\cos t)$ [coordinate wise differentiation].

Hence, this is the velocity of the curve.

✓ 1

(1) Because the domain is $(0, \pi/2)$ the curve traced by γ looks like



Hence, the curve traced by γ lies on an ellipse defined by $\frac{x^2}{3^2} + \frac{y^2}{2^2} - 1 = 0.$

(3) We define the arc-length function

$$l(t) = \int_{t_0}^{t} \|\gamma'\| dt$$

$$= \int_{t_0}^{t} \sqrt{3^2 \sin^2 t + 2^2 \cos^2 t} \, dt$$

$$= \int_{t_0}^{t} \sqrt{9 - 5 \cos^2 t} \, dt$$

From inverse function theorm, as $l'(t) > 0$, l^{-1} exists and is smooth (because γ is regular).

Now, $\tilde{\gamma} := \gamma \circ l^{-1}$

we have
$$\tilde{\gamma}'(t) = \gamma'(l^{-1}(t)) \, (l^{-1})'(t)$$
$$= \frac{\gamma'(l^{-1}(t))}{\|\gamma'(l^{-1}(t))\|}$$

Hence. $\|\tilde{\gamma}'(t)\| = 1$, hence $\tilde{\gamma}$ is a unit-speed parameterization.

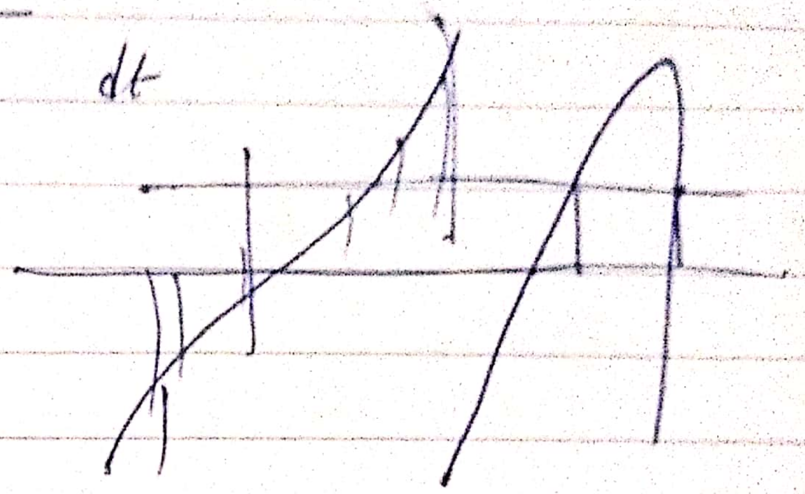
2

$$\gamma \xrightarrow{s^{-1}} \tilde{\gamma} = \gamma \circ s^{-1}$$

regular smooth

$$l(t) = \int_{t_0}^{t} \sqrt{9(1-\cos^2 t) + 4\cos^2 t}\, dt$$

$$= \int_{t_0}^{t} \sqrt{9 - 5\cos^2 t}\, dt$$



Quiz

$\dot{\gamma}(t)$:

$$\boxed{s(t) = \int_{t_0}^{t} \|\gamma'\|\, dt} \qquad s' = \|\gamma'\| > 0$$

$$\boxed{s'(t) = \|\gamma'\|(t)}$$

$$(s^{-1})'(t) = \qquad (s \circ s^{-1})(t) = t$$

$$\Rightarrow s'(s^{-1}(t))\,(s^{-1})'(t) = 1$$

$$\Rightarrow (s^{-1})'(t) = \frac{1}{s'(s^{-1}(t))}$$

$$\tilde{\gamma} := \gamma \circ s^{-1}$$

$$\tilde{\gamma}' = \frac{\gamma'(s^{-1}(t))}{s'(s^{-1}(t))} \cdot (s^{-1})'(t)$$

$$s'(s^{-1}(t)) \to \|\gamma'\|(s^{-1}(t))$$

$$\|\tilde{\gamma}'\| = \frac{\|\gamma'(s^{-1}(t))\|}{\|\gamma'(s^{-1}(t))\|} = 1.$$

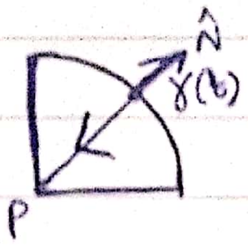
What does a ~~spheric~~ curve of a sphere have? $\gamma'' = \frac{\|\gamma'\|^2}{c}$

$\|\tilde{\gamma} - p\| = \text{const.}$

$\Rightarrow (\tilde{\gamma} - p) \cdot \hat{T} = 0$

$\Rightarrow (\tilde{\gamma} - p) \cdot T' + \hat{T} \cdot \hat{T} = 0$, with $T' = \kappa N$

$\kappa = \frac{\gamma''}{\|\gamma''\|}$



$\Rightarrow (\tilde{\gamma} - p) \cdot \hat{N} = \frac{-1}{\kappa}$

$\Rightarrow (\tilde{\gamma} - p) \cdot \hat{N}' + \cancel{(\tilde{\gamma}') \cdot \hat{N}}^{\,0} = \frac{d}{dt}\left(-\frac{1}{\kappa}\right)$

$\Rightarrow \hat{N}' = -\kappa \hat{T} + \tau \hat{B}$

$\|\tilde{\gamma}\| = \text{const.}\ c \Rightarrow \tilde{\gamma}' \cdot \tilde{\gamma} = 0 \Rightarrow \tilde{\gamma}'' \cdot \tilde{\gamma} = -1$

$\vec{q} = \tilde{\gamma}'' - (\tilde{\gamma}'' \cdot \tilde{\gamma})\tilde{\gamma}$, with $\tilde{\gamma}'' \cdot \tilde{\gamma} = -1$

~~$\vec{q}' = \tilde{\gamma}''' - (\tilde{\gamma}'' \cdot \tilde{\gamma})\tilde{\gamma}' - \tilde{\gamma}'(\tilde{\gamma}''' \cdot \tilde{\gamma})$~~

$= \tilde{\gamma}'' + \tilde{\gamma}$

$\Rightarrow$

~~$\tilde{\gamma}'' \cdot \hat{T}$~~ $\tilde{\gamma}'' = \kappa \hat{N}$

$\tilde{\gamma}$ [illegible] $\|\tilde{\gamma}\| = c$

~~$\tilde{\gamma} = c\hat{N} + \overset{0}{?}\hat{T} + \hat{B}$~~ ?

~~$\tilde{\gamma} =$~~

?

(Thm) Given two smooth functions $\bar{\kappa}:(a,b)\to\mathbb{R}$, $\bar{\kappa}>0$, and $\bar{\tau}:(a,b)\to\mathbb{R}$,
$\exists$ a curve, $\gamma:(a,b)\to\mathbb{R}^3$ so that

$$\kappa[\gamma]=\bar{\kappa}$$
$$\tau[\gamma]=\bar{\tau}$$

If $\exists$ two such curves γ_1, γ_2, with the same curvature and tortion, then $\exists$ a isometry R s.t. $R(\gamma_1(s))=\gamma_2(s)$

* Uniqueness of diff equations

* If $\hat{T}'(t)=$ known $\xrightarrow{\text{then}}$ $\tilde{\gamma}(t)$ unique upto translation.

?
(clever!)

* NOW
Look for

$$\left\{\begin{aligned} e_1'(t) &= \bar{\kappa}e_2 \\ e_2'(t) &= -\bar{\kappa}e_1+\bar{\tau}e_3 \\ e_3'(t) &= \tau e_3 \end{aligned}\right.$$

and $\{e_i(t)\}$ are ON at all t.

$\exists$ solution and they unique(?)

and then $\int e_1(t)\longrightarrow\tilde{\gamma}$

$$y'(t):=(e_1\cdot e_2)'=e_1'\cdot e_2+e_1\cdot e_2'$$
$$=\bar{\kappa}\,\cancel{e_2 e_2}\;\bar{\kappa}\,\cancel{e_1}e_1+\bar{\tau}e_3\cdot e_1$$

Hence for $y(0)=0$, $\exists$ unique sol. $y(t)=0$!

23 SEP 22

$\Rightarrow$ Given any two smooth functions

$$\bar{\kappa} : (a,b) \to \mathbb{R}_{\geq 0}$$
$$\bar{\tau} : (a,b) \to \mathbb{R},$$

then ① $\exists$ a $\gamma(t) : (a,b) \to \mathbb{R}^3$, st.

$$\kappa[\gamma] = \bar{\kappa}, \quad \tau[\gamma] = \bar{\tau}$$

② If $\exists$ two curves γ_1, γ_2 like that then $\exists R \in \mathrm{Iso}(\mathbb{R}^3)$ s.t.

$$\underline{R(\gamma_1) = \gamma_2}.$$

geometry completely det. by $\boxed{\kappa, \tau}$.

proof Claim: find $\bar{T}, \bar{N}, \bar{B}$ so that

$$\left.\begin{aligned} \dot{\bar{T}} &= \bar{\kappa} \bar{N} \\ \dot{\bar{N}} &= -\bar{\kappa}\bar{T} + \bar{\tau} \bar{B} \\ \dot{\bar{B}} &= -\bar{\tau} \bar{N} \end{aligned}\right\} \longrightarrow (\text{Eq } 1)$$

By theory of diff eqn, $\exists$ $\bar{T}, \bar{N}, \bar{B}$ satisfying $\boxed{\text{Eqn } 1}$, given that

$$\bar{T}\big|_{t=t_0} = e_1, \quad \bar{N}\big|_{t=t_0} = e_2, \quad \bar{B}\big|_{t=t_0} = e_3 := e_1 \times e_2$$

Check: If $\{e_i\}$ was a ON basis, then check if $\bar{T}(t), \bar{N}(t), \bar{B}_0(t)$ are all t.

$$f_1(t) := \overline{T}(t) \cdot \overline{T}(t) \quad , \quad f_1(t_0) = 1$$

$$f_1'(t) = 2\dot{\overline{T}}(t) \cdot \overline{T}$$

$$= 2\,(\overline{\kappa}\overline{N}) \cdot \overline{T}$$

$$= 2\,\overline{\kappa}\,\underbrace{\overline{N} \cdot \overline{T}}$$

$$\boxed{f_1'(t) = 2\,\overline{\kappa}\, f_4(t)}$$

$$\frac{d}{dt}\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{bmatrix}(t) = \begin{bmatrix} 2\overline{\kappa} f_4 \\ \\ \\ \\ \\ \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}(t) \text{ satisfy it.}$$

$$f_2(t) := \overline{N}(t) \cdot \overline{N}(t)$$

$$f_2'(t) = 2\,\dot{\overline{N}}(t) \cdot \overline{N}(t)$$

$$=$$

Now, $\gamma(t) = \int \overline{T}(t)\,dt$

Here, $\dot{\gamma}(t) = \overline{T} \Rightarrow \|\dot{\gamma}\| = 1.$

$$\langle \qquad \rangle$$

$$\ddot{\gamma} = \dot{\overline{T}} = \overline{\kappa}\,\overline{N}$$

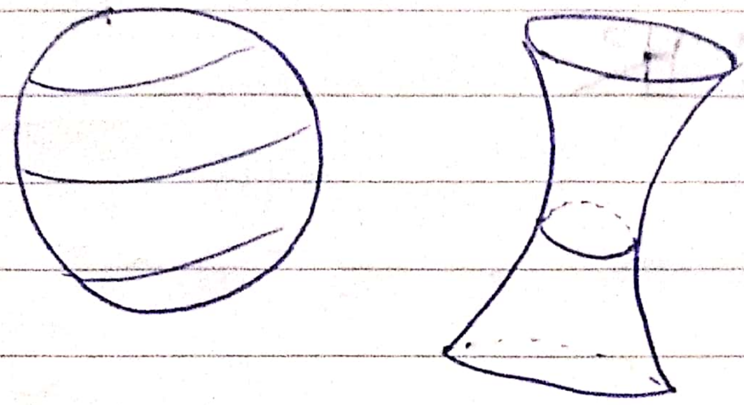
$\Rightarrow \|\ddot{\gamma}\| = \overline{\kappa}$ Hence curvature is what we wanted!

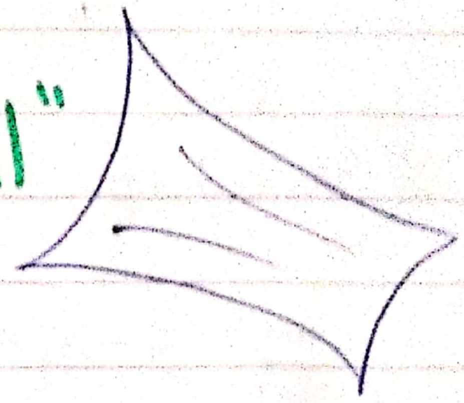
26/SEP/22

Previously, given $\kappa, \tau \Longrightarrow$ you have a curve unique up to isometry. in $\mathbb{R}^3$

Hence curves is complete!

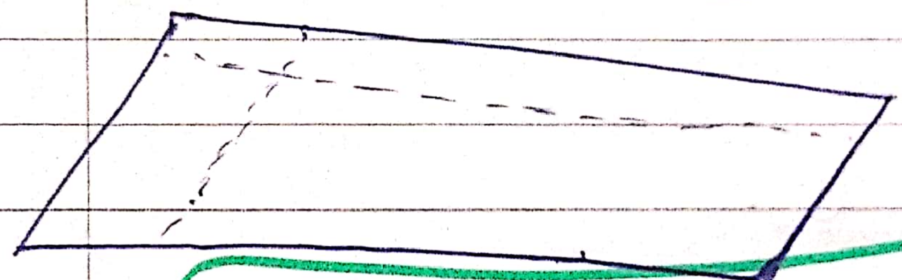
SURFACES





"two dimentional"

define dimention!



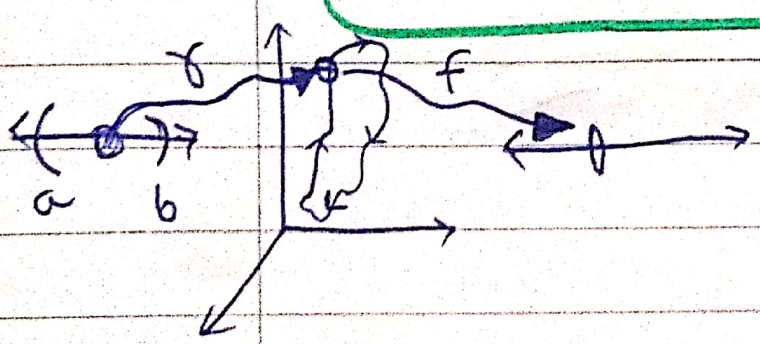
$f : \mathbb{R}^3 \longrightarrow \mathbb{R}$; $\gamma : (a,b) \longrightarrow \mathbb{R}^3$

$t \mapsto \begin{bmatrix} \gamma_x(t) \\ \gamma_y(t) \\ \gamma_z(t) \end{bmatrix}$

Thm

$$\frac{d}{dt}(f \circ \gamma)(t)$$

$$= \frac{df}{dt}(\gamma(t)) = \partial_x f \begin{bmatrix} \gamma_x(t) \\ \gamma_y(t) \\ \gamma_z(t) \end{bmatrix} \gamma_x'(t) + \partial_y f \begin{bmatrix} \gamma_x(t) \\ \gamma_y(t) \\ \gamma_z(t) \end{bmatrix} \gamma_y'(t)$$



$$+ \partial_z f \begin{bmatrix} \gamma_x(t) \\ \gamma_y(t) \\ \gamma_z(t) \end{bmatrix} \gamma_z'(t).$$

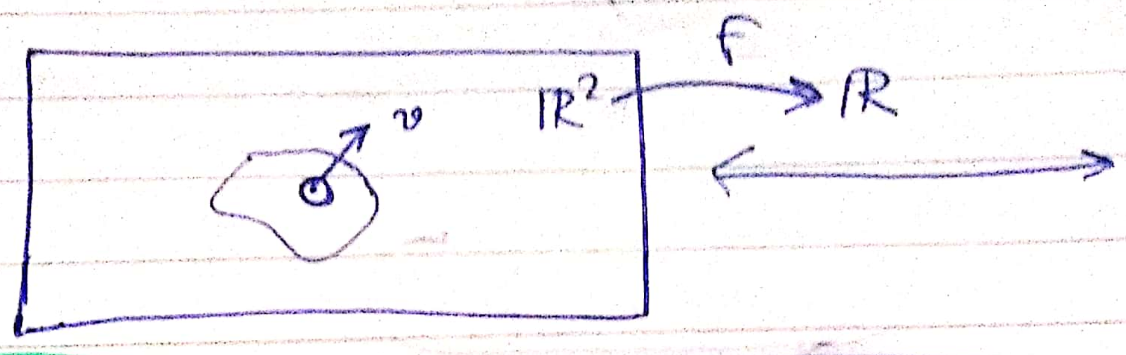
$$\cancel{= (\nabla f) \cdot \gamma'(t) = (\text{grad} f) \cdot \gamma'(t)}$$

chain rule

$$= (\text{grad} f)(\gamma(t)) \cdot \gamma'(t)$$

"$\frac{\partial f}{\partial v}(x)$"

~~(Defn) Given $f: \mathbb{R}^3 \to \mathbb{R}$ and vector $v \in \mathbb{R}^3$,~~ U



(Defn) A subset $U \subseteq \mathbb{R}^n$ is called $\boxed{\text{open}}$, if for any $p \in U$, $\exists$ a ball

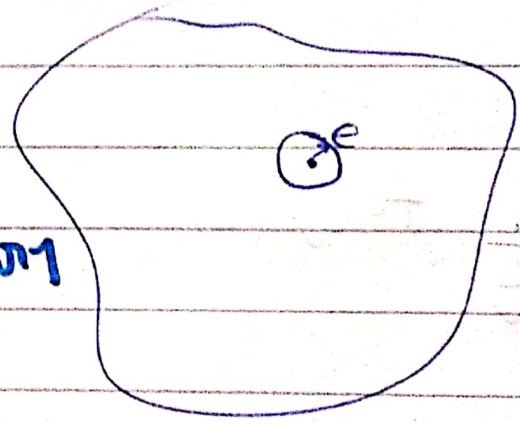
$$\epsilon\text{-}B(p) := \{ q \in \mathbb{R}^n : \|q - p\| < \epsilon \}$$

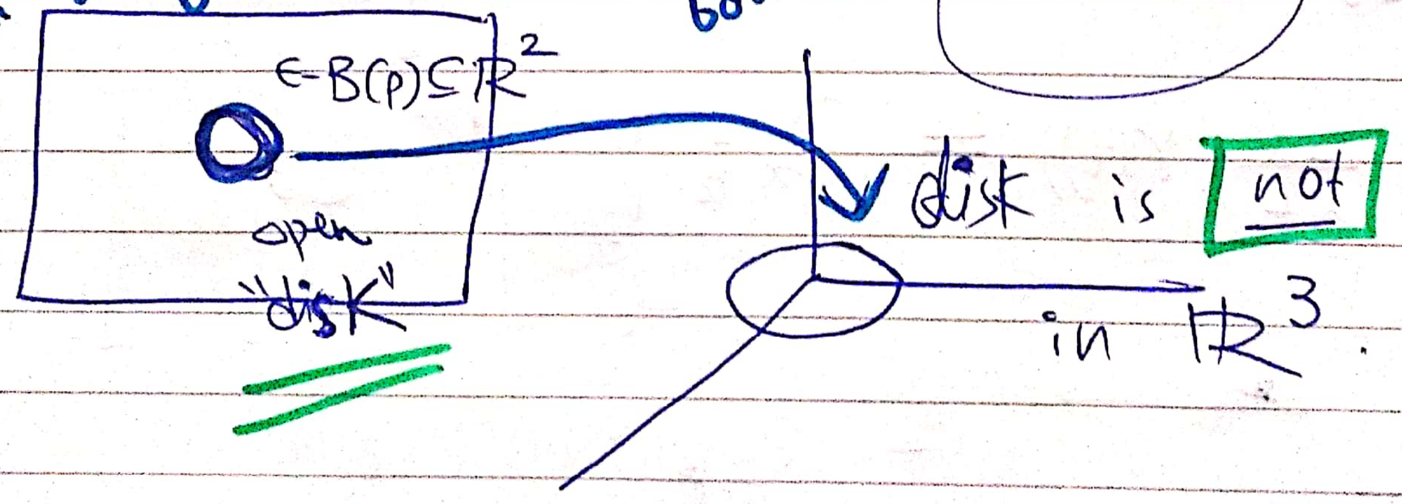
such that

$$\epsilon\text{-}B(p) \subseteq U.$$

* open sets of $\mathbb{R}^2$ are "disks" w/out boundary

open sets of $\mathbb{R}^3$ are "volumes" without boundary





disk is $\boxed{\text{not}}$ open in $\mathbb{R}^3$.

[Chain rule works for $U \subseteq \mathbb{R}^3$ be open and $f: U \to \mathbb{R}$, etc...]

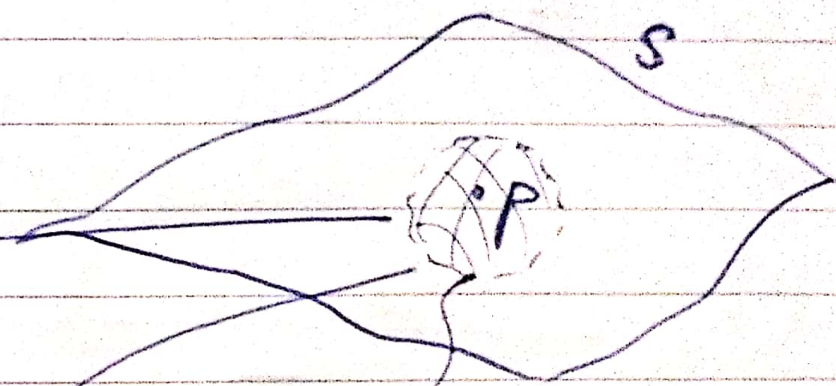
open $\subseteq \mathbb{R}^n$

(Defn) Given $f: U \to \mathbb{R}$ is <u>smooth</u> if all its partial derivatives exist.

9 SEP 22

How to define surfaces?

* <u>Surfaces at first are $S \subseteq \mathbb{R}^3$</u>
* <u>consider $p \in S$</u>



"pach"

$S \cap V$

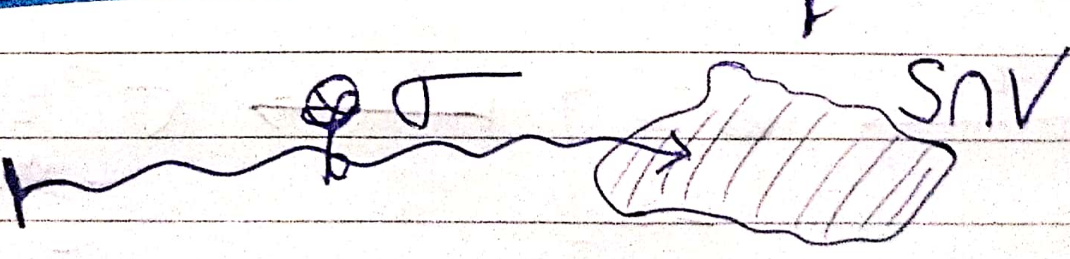
where V is a open subset of $\mathbb{R}^3$.

is a "open subset of S".

(Thm) No complete, nice? chart (?) to map all surface of a sphere.

Defn

subset of $\mathbb{R}^2$



$U \subseteq \mathbb{R}^2$ (open) and choose $\sigma \cancel{\varphi}: U \longrightarrow S \cap V$

σ be a local chart

① σ is bijective. $\Rightarrow \exists\, \varphi^{-1}$

② $\sigma \in C^\infty(U)$ (smooth)

~~③ $\sigma^{-1} \in C^\infty(S \cap V)$~~

③ σ^{-1} exists and is continuous.

④ σ is regular

$\sigma\begin{bmatrix}x\\y\end{bmatrix} = \begin{matrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{matrix}$ $\qquad \dfrac{\partial\sigma}{\partial x} \times \dfrac{\partial\sigma}{\partial y} \neq 0$

for all $\begin{bmatrix}x\\y\end{bmatrix} \in U$.

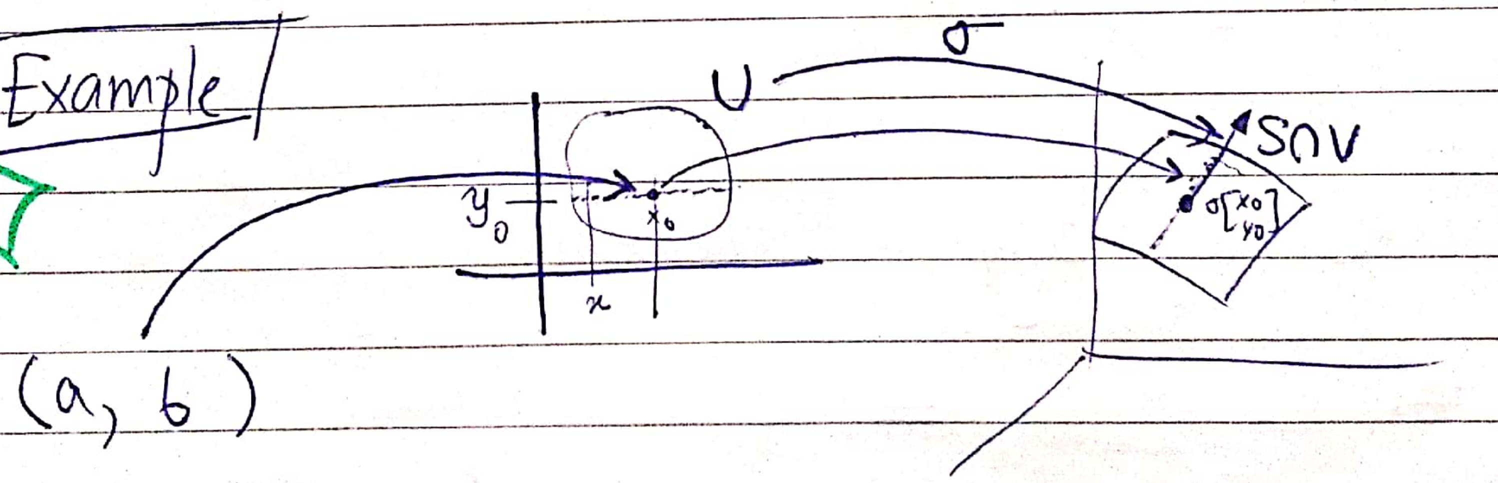
define surface by patches

* σ is called a surface patch, that "covers" $S \cap V$.

curves on surfaces

(Defn) $\vec{V}$ is a $\boxed{\text{tangent}}$ vector at $p \in S \subseteq \mathbb{R}^3$ (surface)
if $\exists$ $\gamma : (a,b) \longrightarrow S \subseteq \mathbb{R}^3$
s.t. $p = \gamma(t_0)$
and $V = \gamma'(t_0)$

Example



$$t \longmapsto \begin{bmatrix}t\\y_0\end{bmatrix} \longmapsto \sigma\begin{bmatrix}t\\y_0\end{bmatrix}$$

$\gamma(t)$

and $\gamma'(\cancel{t}) = \left(\dfrac{\partial\sigma}{\partial x}\right)\begin{bmatrix}t_0\\y_0\end{bmatrix}$

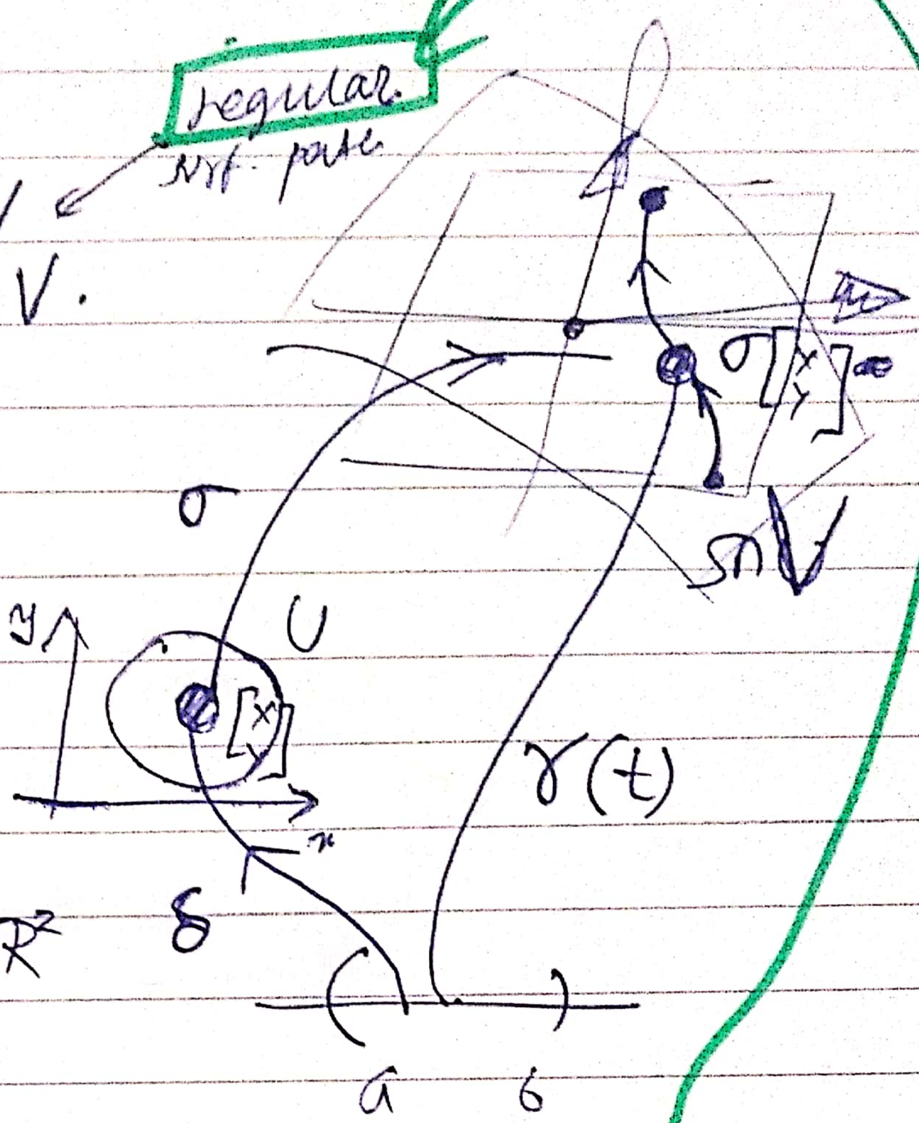
Hence, $\partial_x\sigma, \partial_y\sigma$ is a tangent vector.

Now Take any

$\sigma : U \longrightarrow S \cap V$ regular surf. patch

$\gamma : (a,b) \longrightarrow S \cap V.$

smooth curve.



Now define

$\delta(t) := (\sigma^{-1} \circ \gamma)(t).$

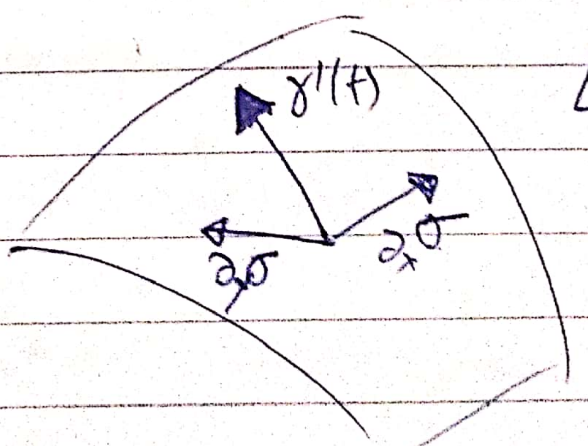
$\delta : (a,b) \longrightarrow U \subset \mathbb{R}^2$

Claim: $\delta(t)$ is smooth (proof later). regular surface

then $\gamma(t) = (\sigma \circ \delta)(t)$

$$\gamma'(t) = \underbrace{\delta_x'(t)\cdot\partial_x\sigma + \delta_y'(t)\cdot\partial_y\sigma}_{\text{L.C. of } \partial_x\sigma \text{ and } \partial_y\sigma}$$

tangent

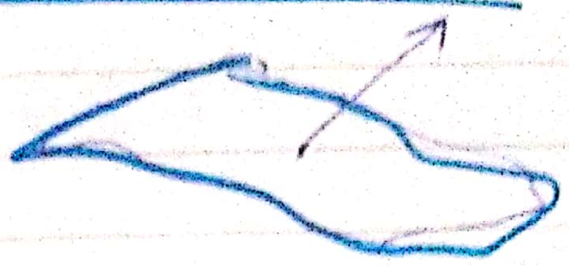


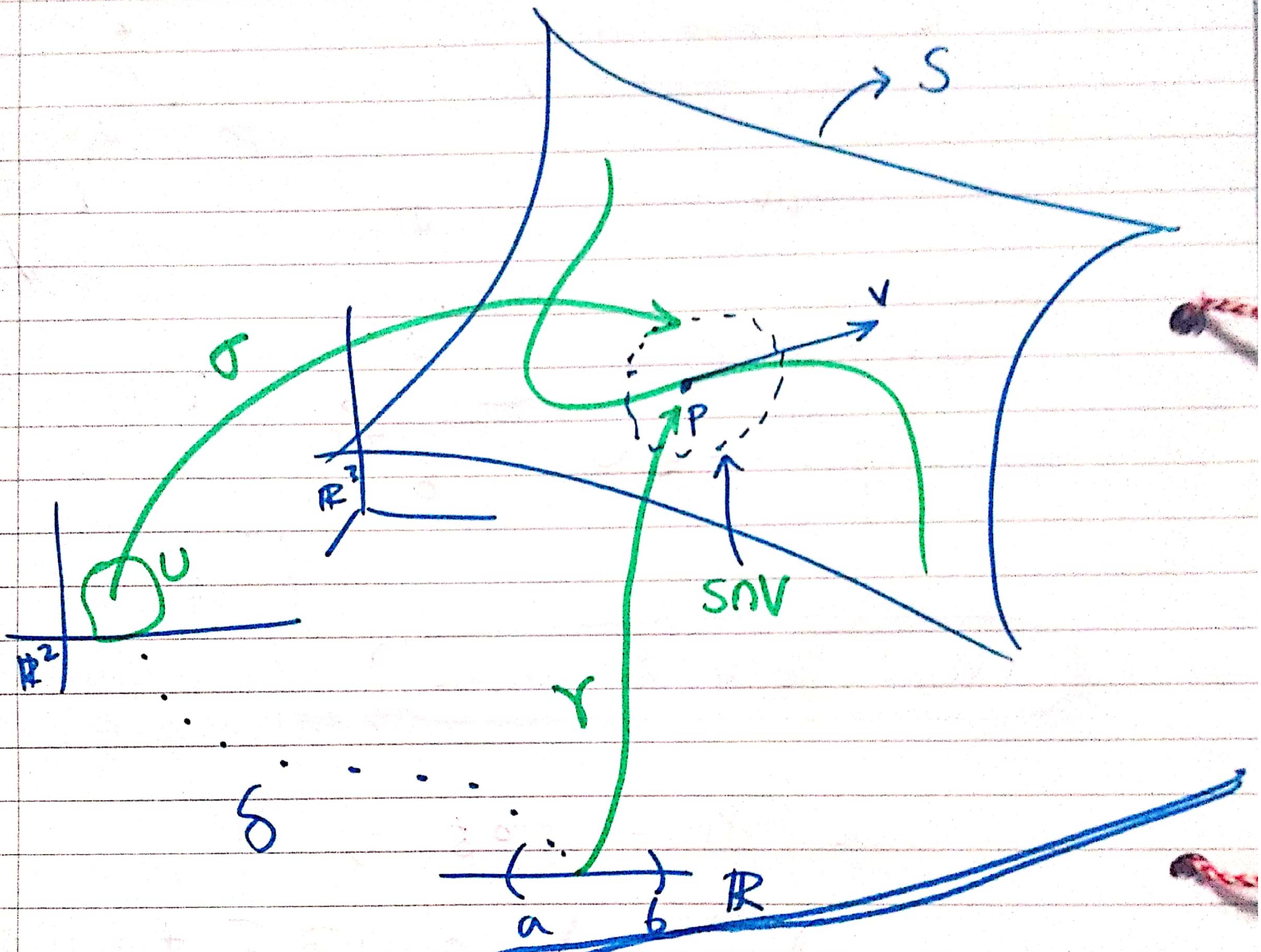
Hence, $\partial_x\sigma$, $\partial_y\sigma$ span $\{\text{tangents}\}$.

" necessary to stay on S "

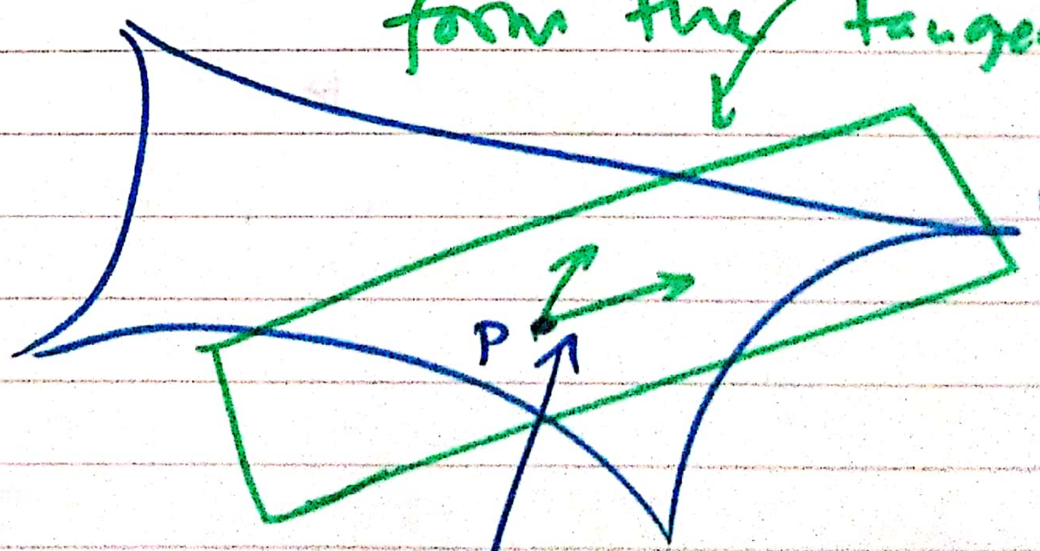
Next

* normal curvature
* geodesic curvature





set of all tangent vectors form the tangent plane at p " T_p "



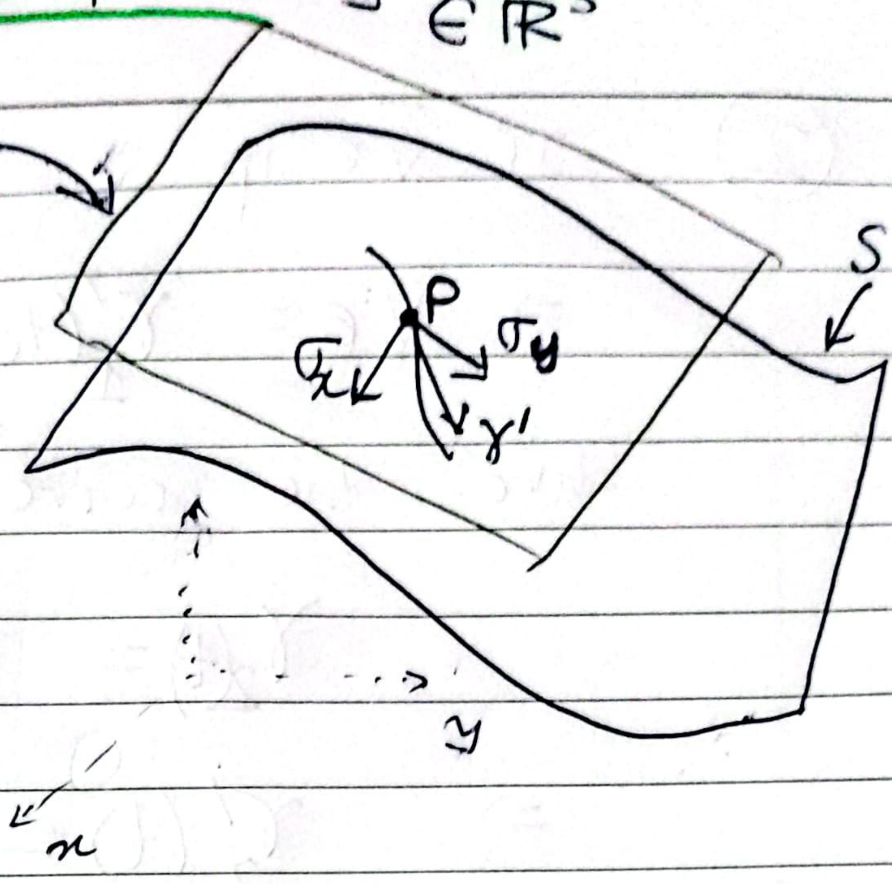
$\{\partial_x \sigma(p), \partial_y \sigma(p)\}$ form a basis of T_p

at p   at p

tangent vector (to surface) at point $p \in S$:= a velocity vector of a curve at point $p \in S$. $\in \mathbb{R}^3$

* $T_p := \{$ set of all tangent vectors $\}$

by changing curves

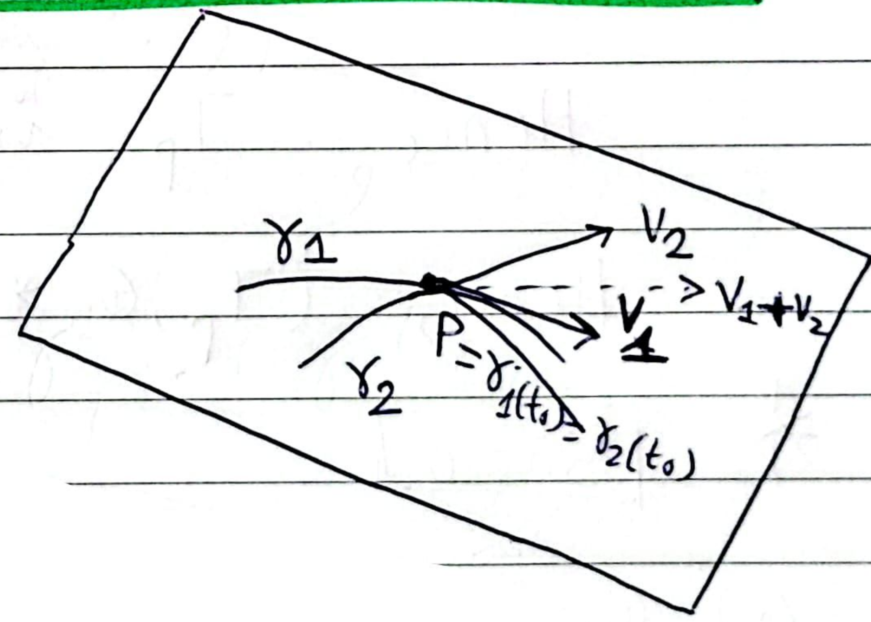


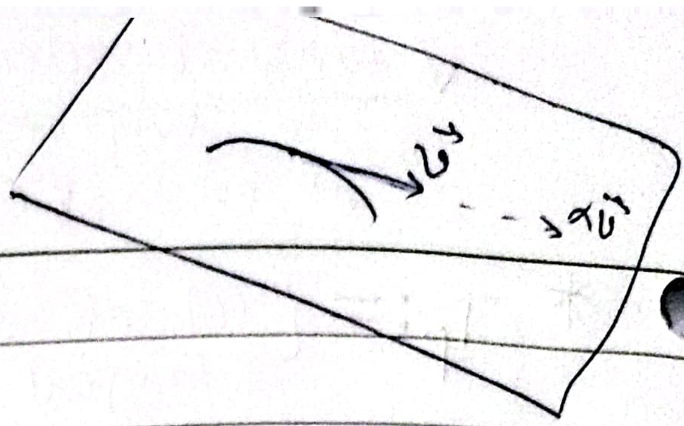
* $T_p \subseteq \mathbb{R}^3$ because the tangent vectors $\in \mathbb{R}^3$.

* Take $(T_p, \oplus, \circledast)$

vector operations from the vector space $(\mathbb{R}^3, +, *)$

* Show : $(T_p, \oplus, \circledast)$ is a vector space. (subspace of $\mathbb{R}^3$).





(2) If $\vec{v} \in T_p$

$\Rightarrow v = \gamma_1'(t_0)$, $p = \gamma_1(t_0)$

take the curve

$$\gamma_2(t) = \gamma_1(\alpha t)$$

$$\Rightarrow \quad \gamma_2'(t) = \alpha\, \gamma_1'(\alpha t)$$

Hence, $\gamma_2(t_0) = \cancel{p}\ \gamma_1(\alpha t_0) = p$

and $\gamma_2'(t_0) = \alpha\, \gamma_1'(\alpha t)$

$= \alpha v$

Hence, $\alpha \vec{V} \in T_p$.

Hence, T_p is closed in $\oplus$ and $\circledast$.

Hence, $(T_p, \oplus, *)$ is a vector space.

* $\boxed{\text{Show}}$

(1) any $\vec{V} \in T_p$ is a linear combination of σ_x, σ_y

$\Rightarrow \vec{V} = \lambda_1 \sigma_x + \lambda_2 \sigma_y$

(2) any $b_1 \sigma_x + b_2 \sigma_y$ $\in T_p$

(1) $\Rightarrow \quad T_p \subseteq \operatorname{span}(\sigma_x, \sigma_y)$

(2) $\Rightarrow \quad \operatorname{span}(\sigma_x, \sigma_y) \subseteq T_p$

$\Rightarrow \quad \boxed{T_p = \operatorname{span}(\sigma_x, \sigma_y)}$

and $\sigma_x \times \sigma_y \neq \vec{0} \Rightarrow \sigma_x, \sigma_y$ are linearly independent

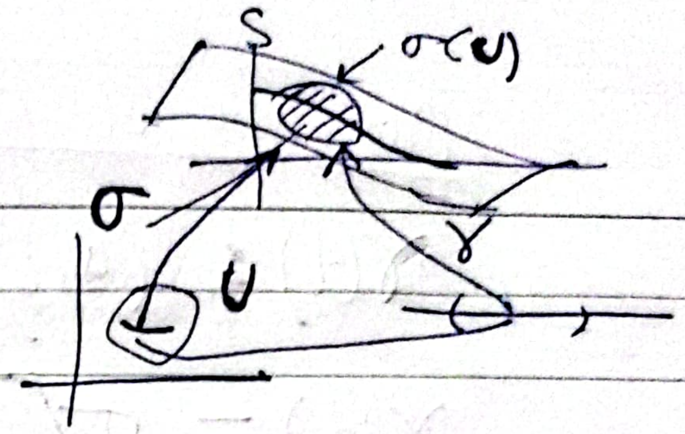
Hence,

$\boxed{\{\sigma_x, \sigma_y\} \text{ form a basis of } T_p.}$

$\sigma_x(p), \quad \sigma_y(p).$

$\boxed{\text{Hence, } \dim(T_p) = 2}$!

Recall

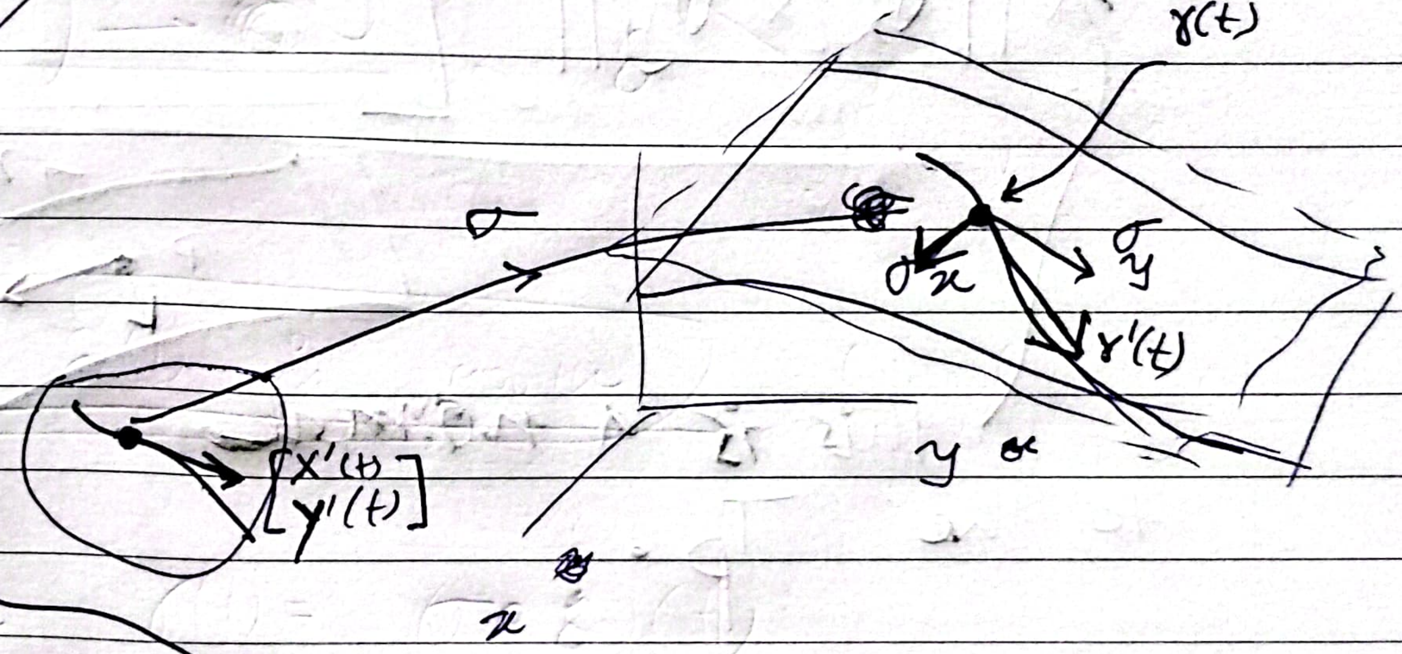


$\gamma : (a, b) \longrightarrow S \subseteq \mathbb{R}^2$ surface

Say $\gamma(t) \in \sigma(U)$ for some (U, σ) regular chart. smooth.

$\Rightarrow \gamma(t) = \sigma\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$

$\gamma'(t) = x'(t)\,\sigma_x\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + y'(t)\,\sigma_y\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$



* any tangent vector $\gamma'(t_0)$ ⟷ a linear combination of σ_x σ_y

$v = a\sigma_x + b\sigma_y$

have a curve with

$x' = a,\ y' = b$;

take a line in U.

Curvature of curves on surface

$$\gamma(t) : (a,b) \longrightarrow \underset{\text{(surface)}}{S} \subseteq \mathbb{R}^3$$ which is unit speed.

$$\gamma(t) = \sigma\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

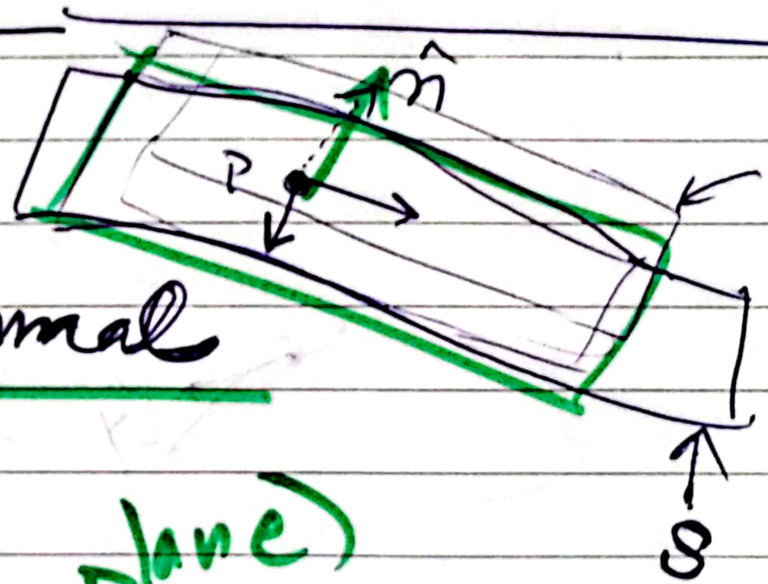
So, $$\|\gamma''(t)\| =: \kappa(t)$$

$T_p := \{\text{tangent vectors at } p\} \subseteq \mathbb{R}^3$

$*$ T_p is a 2-dim vector space $(T_p, \oplus, \circledast)$

and $\{\sigma_x, \sigma_y\}$ form a basis.

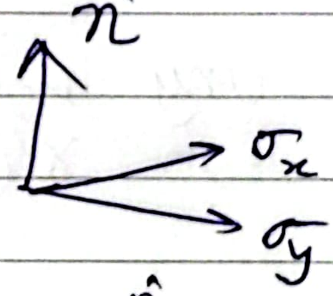
$$\boxed{\hat{n} := \frac{\sigma_x \times \sigma_y}{\|\sigma_x \times \sigma_y\|}}$$



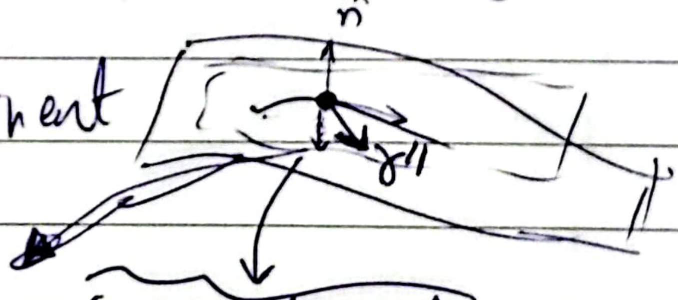
this is a normal to the T_p.

(T_p is a plane)

$$\gamma' = x'\sigma_x + y'\sigma_y$$



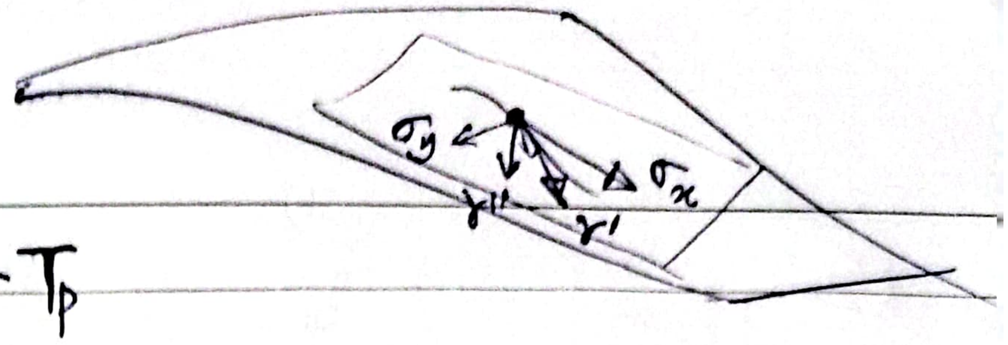
Take γ'' but only the component along T_p

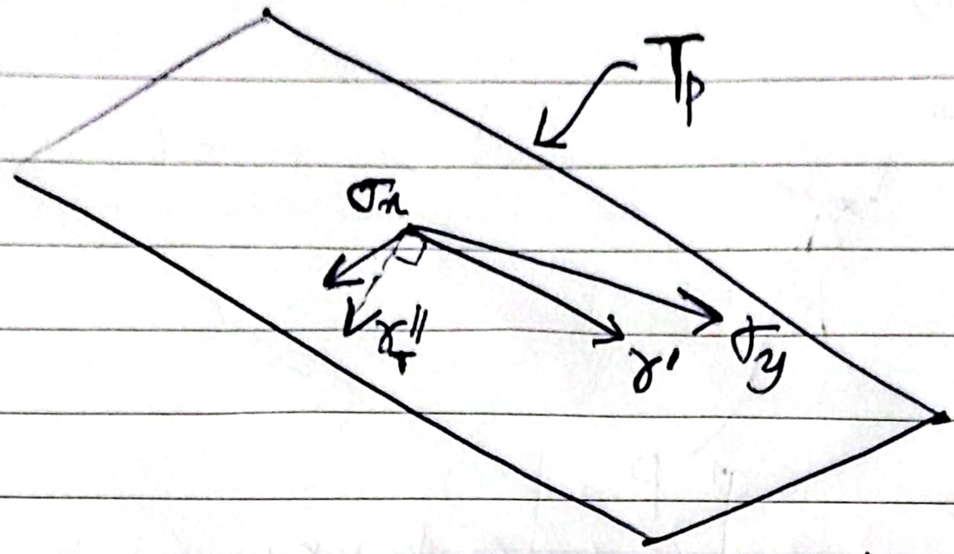


Now

$$\gamma''_T := \gamma'' - (\gamma'' \cdot \hat{n})\hat{n}$$

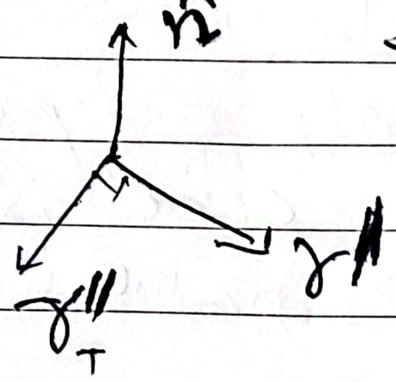
$*$ $\gamma''_T \in T_p$





and $\boxed{\gamma''_T \cdot \gamma' = 0}$

and of course $\boxed{\gamma''_T \cdot \hat{n} = 0.}$



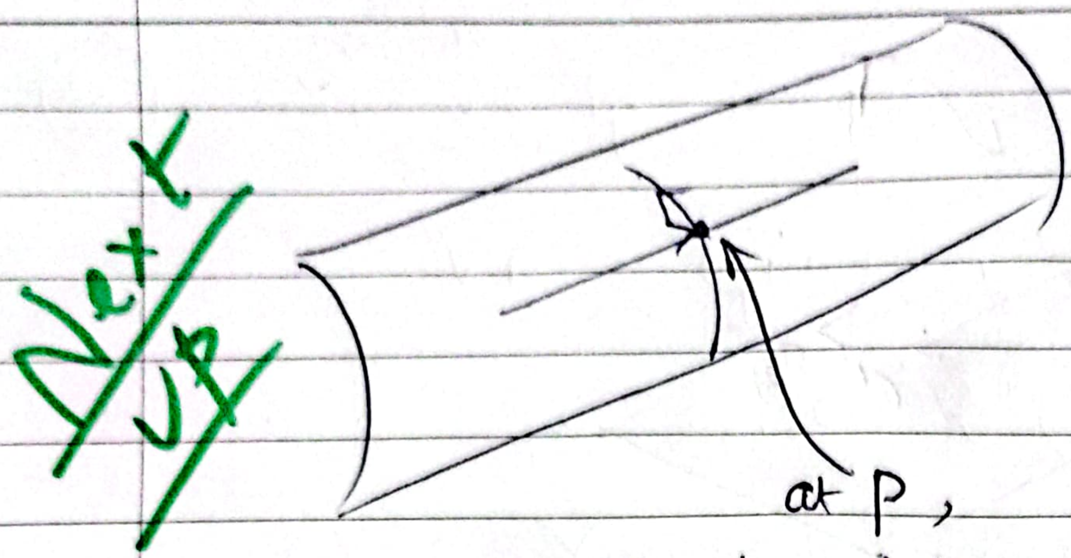
$\therefore$ γ''_T is along $\gamma' \times \hat{n}$.

Now ① $\boxed{\kappa_n(t) := \gamma'' \cdot \hat{n}}$ (normal curvature)

② $\boxed{\kappa_g(t) := (\gamma''_T) \cdot (\gamma' \times \hat{n})}$ (geodesic curvature).

$\therefore \gamma'' = \kappa_n \hat{n} + \kappa_g \gamma' \times \hat{n}$

$\Rightarrow \boxed{\|\kappa\|^2 = \|\kappa_n\|^2 + \|\kappa_g\|^2}$.



at P,

we have two curves having different K_n.

So K_n depends on curves.

Claim: K_n only depends on the direction the curve is moving along with the point on the surface.

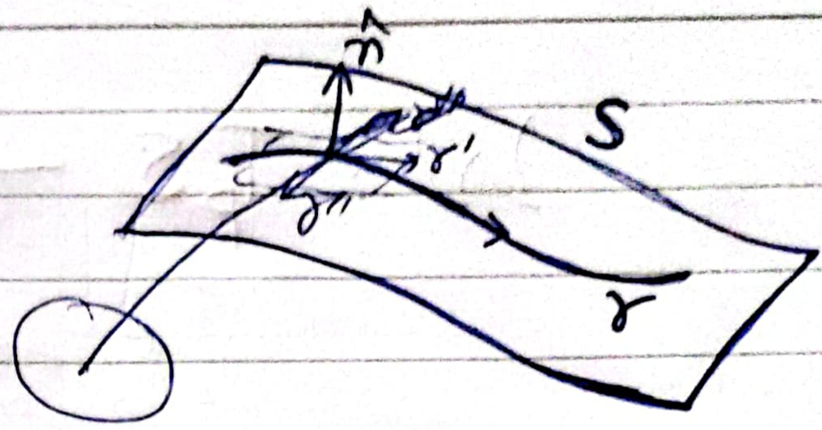
$\begin{Bmatrix} \text{position} \\ \text{direction} \end{Bmatrix} \longrightarrow K_n$

$K_n : S \times T_p \longrightarrow \mathbb{R}$?

$\gamma : I \longrightarrow S$ 14 OCT 22

$$\kappa^2 = \kappa_n^2 + \kappa_g^2$$

(usual) (normal) (geodesic)



$$\kappa_n := \gamma'' \cdot \hat{n}$$

$$* \quad \underbrace{\frac{d}{dt}(\gamma' \cdot n)}_{\vec{0}} = \gamma'' \cdot \hat{n} + \gamma' \cdot \frac{d}{dt}\hat{n}(\gamma(t))$$

$\hat{n} : S \longrightarrow \mathbb{R}^3$

$\tilde{n} : U \longrightarrow \mathbb{R}^3$

$\tilde{n} := \hat{n} \circ \sigma$

$$\Rightarrow \underbrace{\gamma'' \cdot \hat{n}}_{\kappa_n} = - \gamma' \cdot \frac{d\hat{n}(\gamma(t))}{dt}$$

$$\cancel{= - \begin{bmatrix} \gamma'_x \\ \gamma'_y \end{bmatrix} \cdot \left(\partial_x n \, \gamma'_x + \partial_y n \, \gamma'_y \right)}$$

$$\cancel{= -\left(\partial_x \hat{n} \cdot \hat{x} \, (\gamma'_x)^2 + \partial_y \hat{n} \cdot \hat{y} \, (\gamma'_y)^2 \right)}$$

~~Hence, κ_n only depends on $\begin{bmatrix} \gamma_x \\ \gamma_y \end{bmatrix}$, $\begin{bmatrix} \gamma'_x \\ \gamma'_y \end{bmatrix}$.~~

we cannot do derivative of functions like $f : S \longrightarrow \mathbb{R}^3$

$\delta(t) := \sigma^{-1}(\gamma(t))$

$$= - \gamma' \cdot (\underbrace{\hat{n} \circ \sigma}_{} \circ \delta)'(t)$$

$$(\hat{n} \circ \sigma)\begin{bmatrix} x \\ y \end{bmatrix} =: \tilde{n}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= - \gamma' \cdot (\tilde{n} \circ \delta)'(t)$$

$$= \underbrace{- \gamma' \cdot \left(\partial_x \tilde{n} \, \delta'_x + \partial_y \tilde{n} \, \delta'_y \right)}_{\text{only depends on point and velocity of curve.}}$$

$$\exists \; W : \underbrace{T_p S}_{\text{tangent plane at } p \text{ of surface } S} \longrightarrow \mathbb{R}^3$$

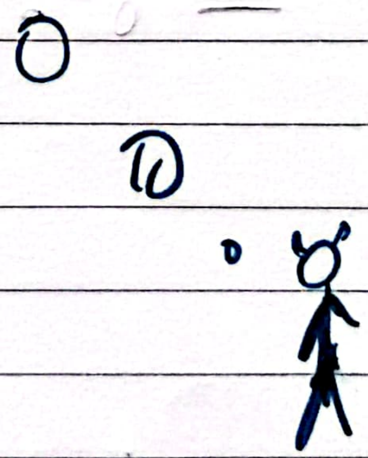
such that

$$W(\gamma'(t_0)) = \frac{d}{dt}\hat{n}(\gamma(t))$$

$$\Rightarrow \quad \kappa_n(t) = -\gamma' \cdot W(\gamma'(t)).$$

essentially, W is the total derivative of $\tilde{n}\begin{bmatrix}x\\y\end{bmatrix}$.

$$W = d\tilde{n} \, ?$$



(I'm pretty sure about this:)

$$\mathcal{W}(\gamma'(t)) = \underbrace{(\tilde{n} \circ \gamma)'(t)}$$

$$= \partial_x \tilde{n}\, \delta_x' + \partial_y \tilde{n}\, \delta_y'$$

$$= [\partial_x \tilde{n} \quad \partial_y \tilde{n}] \underbrace{\begin{bmatrix} \delta_x' \\ \delta_y' \end{bmatrix}}_{\delta'(t)}$$

$$\mathcal{W}(\gamma'(t)) = [\partial_x \tilde{n} \quad \partial_y \tilde{n}] (\Gamma) \begin{bmatrix} \gamma'(t) \\ \end{bmatrix}$$

$$\Gamma : T_pS \xrightarrow{\sim} \mathbb{R}^2 ; \; \Gamma(\gamma') = \delta'$$

$$\Rightarrow \quad \mathcal{W}(\vec{v}) = \underbrace{\left([\partial_x \tilde{n} \quad \partial_y \tilde{n}] \circ \Gamma \right)}_{\text{"matrix multiplication"}} (\vec{v})$$

$$\gamma' = \sigma_x \delta_x' + \sigma_y \delta_y'$$

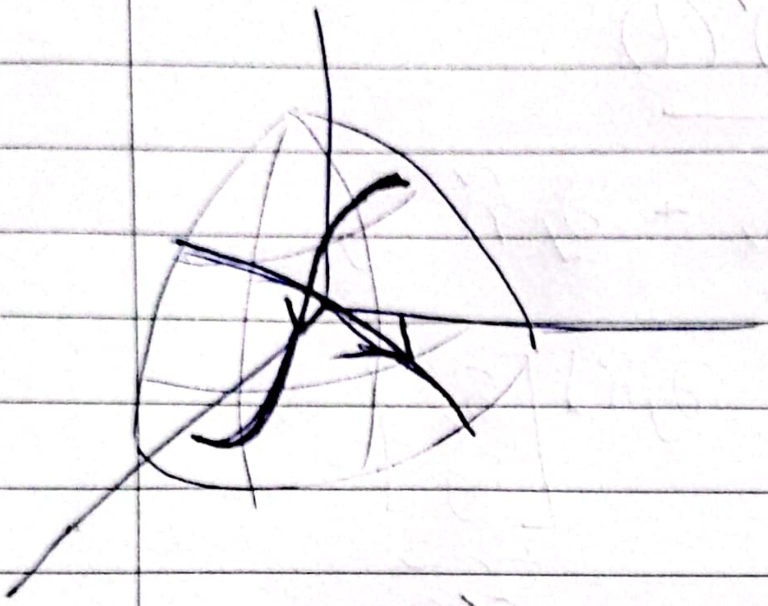
$$= [\sigma_x \quad \sigma_y][\delta']$$

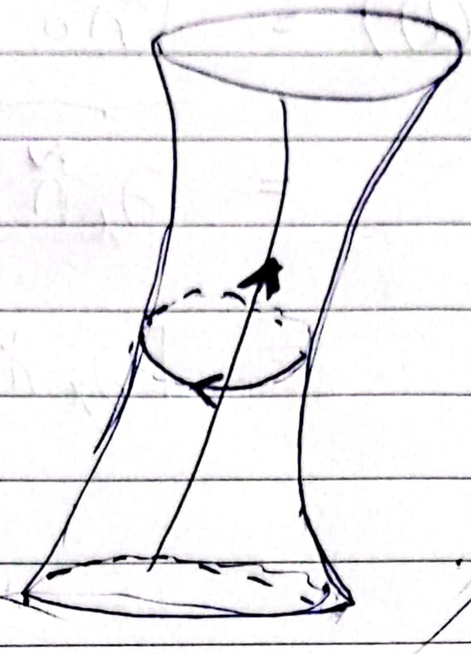
$$\Gamma(\gamma') = \delta'$$

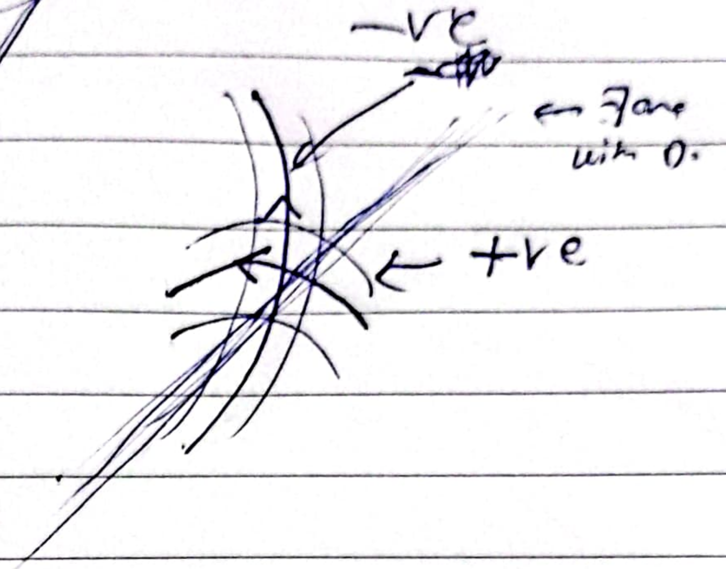
$$\Rightarrow \quad \Gamma(v) = [\sigma_x \quad \sigma_y]^{-1}(v)$$

$$\therefore \quad \mathcal{W} = \underbrace{[\partial_x \tilde{n} \quad \partial_y \tilde{n}]}_{\text{total derivative of } \tilde{n}} \cdot \underbrace{[\sigma_x \quad \sigma_y]^{-1}}_{\text{inverse of total derivative of } \sigma}$$

17-OCT-22

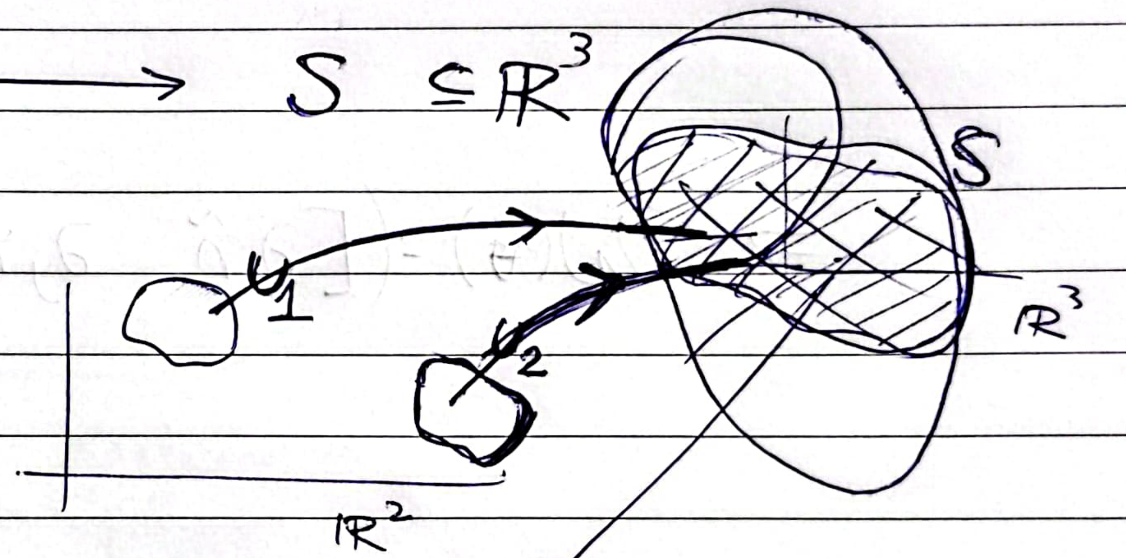




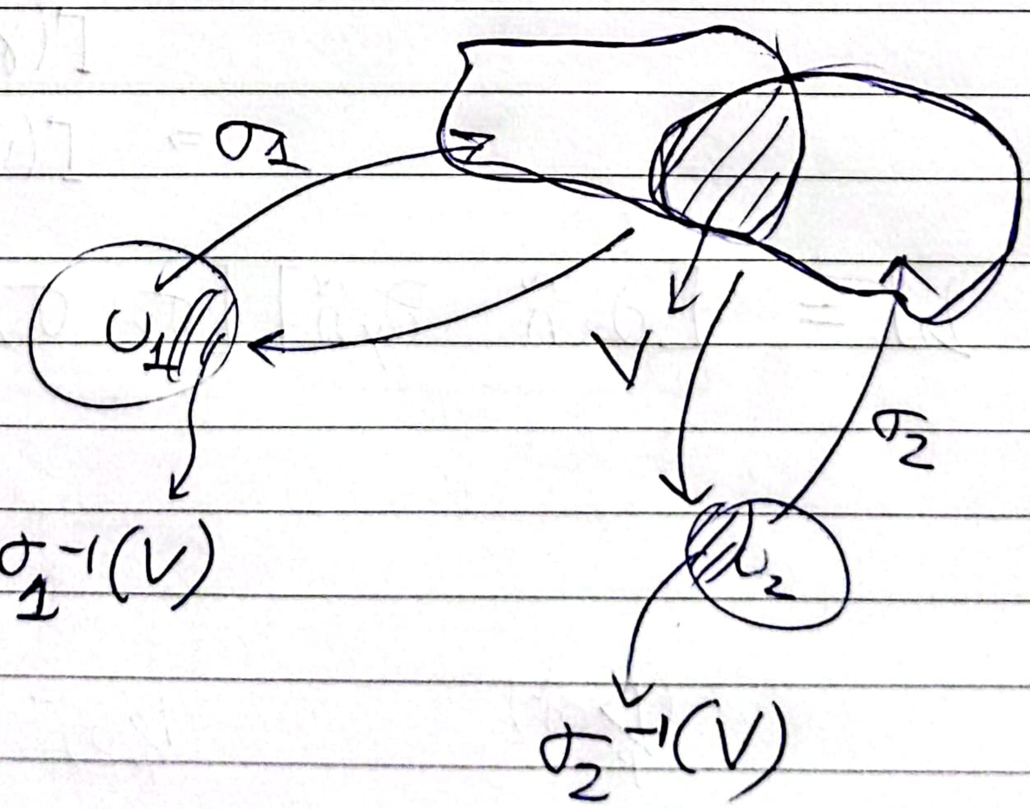


$$\sigma_1 : U_1 \longrightarrow S \subseteq \mathbb{R}^3$$

$$\sigma_2 : U_2 \longrightarrow S \subseteq \mathbb{R}^3$$



$$V := \sigma_1(U_1) \cap \sigma_2(U_2)$$



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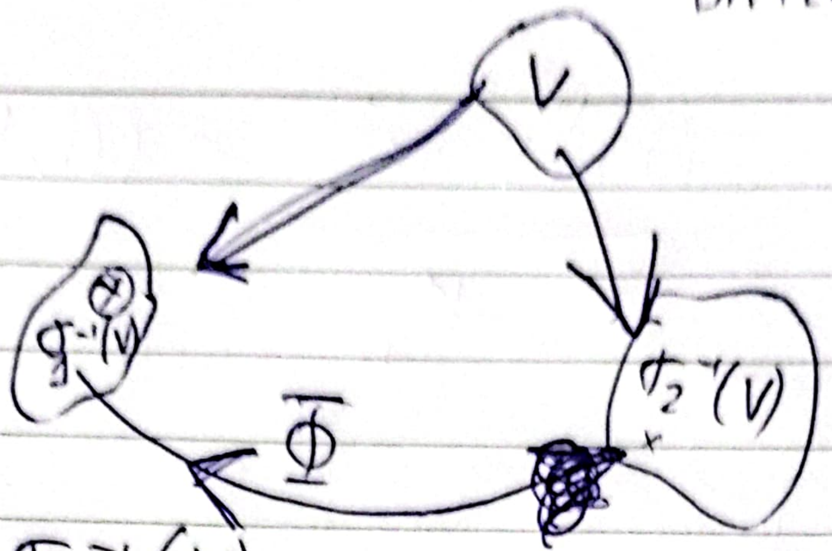


$$\exists\ \Phi : \underset{\subseteq \mathbb{R}^2}{\sigma_2^{-1}(V)} \longrightarrow \underset{\subseteq \mathbb{R}^2}{\sigma_1^{-1}(V)}$$

s.t.

$$\sigma_2\Big|_{\sigma_2^{-1}(V)} = \sigma_1\Big|_{\sigma_1^{-1}(V)} \circ \Phi$$

where Φ should be a diffeom.

(Inverse function them)

$$\boxed{\begin{array}{l} \text{Say } f : U_1 \to U_2 \ ;\ U_1, U_2 \subseteq \mathbb{R}^n ,\ f \in C^\infty . \\ \text{and } p \in U_1 ,\ \underline{\det(J_p(f))} \neq 0 ,\ \text{then} \\ \exists\ V_1 \subseteq U_1 ,\ \exists V_2 \subseteq U_2 ,\ \text{s.t. } p \in V_1 , \\ f\big|_{V_1} : V_1 \to V_2 \text{ is a diffeom.} \end{array}}$$

Page 1/2

MTH201
Quiz-2

12-OCT-22

5/5

Rupadarshi Ray
MS21165

Consider the set

$$S := \{ (x, y, \sqrt{1-x^2-y^2}) \mid -\tfrac{1}{10} < x, y < \tfrac{1}{10}$$

(1) S is a part of a well-known surface. Which surface is that and which part? (2)

(2) Find a regular surface patch for S. Prove that it is regular. (3)

---

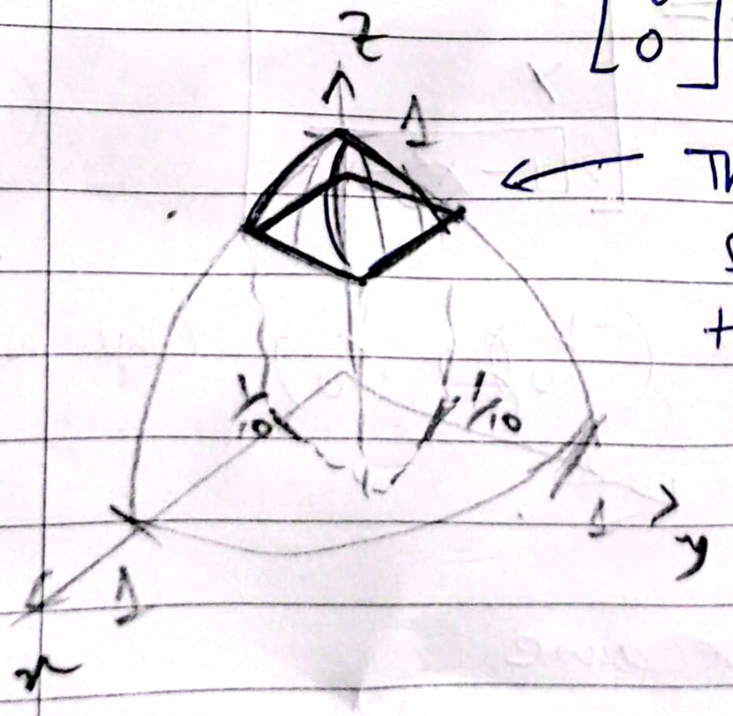
(1) The points in subset $S \subseteq \mathbb{R}^3$ satisfy

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in S \Rightarrow x^2 + y^2 + z^2 = 1$$
and $-\tfrac{1}{10} < x < \tfrac{1}{10}$

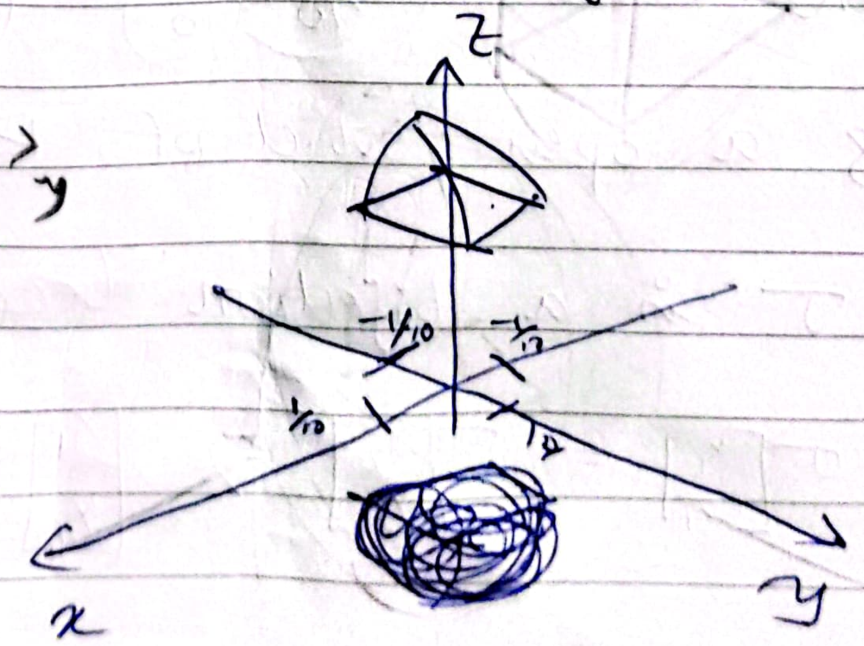
$-\tfrac{1}{10} < y < \tfrac{1}{10}$, $z > 0$

2

Hence, ~~the~~ ~~set~~ S is part of a sphere of radius 1 centered at $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (origin).



The ~~~~ part of S that lies in the 1st quadrant (+, +, +)



Hence, the part of ~~the~~ the sphere (S^2) with radius 1 centered at origin at satisfies $z > \frac{\sqrt{98}}{10}$ ~~[illegible]~~ equals to the set S.

$$\left(\tfrac{1}{10}\right)^2 + \left(\tfrac{1}{10}\right)^2 + z^2 = 1$$

$$\Rightarrow z^2 = 1 - \frac{2}{100}$$

$$z = \sqrt{\frac{98}{100}}$$

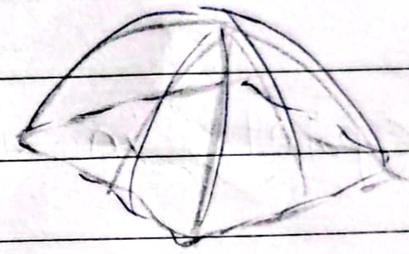
$$= \frac{\sqrt{98}}{10}$$

(2) ~~As S is not connected, we need here two surface patches σ.~~

~~Defining the following:~~

~~σ_+ :~~

~~σ_+~~



$\vec{0}$

(2) We simply define

$$\sigma : \left(-\tfrac{1}{10}, \tfrac{1}{10}\right)^2 \longrightarrow S$$

$$\sigma\begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} x \\ y \\ \sqrt{1-x^2-y^2} \end{bmatrix}$$

I We know $\left(-\tfrac{1}{10}, \tfrac{1}{10}\right) \times \left(-\tfrac{1}{10}, \tfrac{1}{10}\right)$ (the domain) is a open subset of $\mathbb{R}^2$.

II σ is a bijection because

$$\sigma\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \sigma\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

Page 2/2

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and all points of S are mapped onto.

The σ^{-1} is the projection of S onto the x-y plane.

III σ, σ^{-1} are continuous.

IV σ is smooth because $x, y,$ are smooth functions and $\sqrt{1-x^2-y^2}$ at $1-x^2-y^2 \neq 0$ is a smooth function.

V $\sigma_x = \frac{\partial \sigma}{\partial x} = \begin{bmatrix} 1 \\ 0 \\ \frac{\partial}{\partial x}\sqrt{1-x^2-y^2} \end{bmatrix}$, $\sigma_y = \frac{\partial \sigma}{\partial y} = \begin{bmatrix} 0 \\ 1 \\ \frac{\partial}{\partial y}\sqrt{1-x^2-y^2} \end{bmatrix}$

3

σ_x and σ_y are linearly independent because $\begin{bmatrix} 1 \\ 0 \\ \lambda \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ \beta \end{bmatrix}$ are always linearly independent regardless of λ, β, since

$$c_1 \begin{bmatrix} 1 \\ 0 \\ \lambda \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ \beta \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} c_1 \\ 0 \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ c_2 \\ \beta \end{bmatrix}$$

$$\Rightarrow c_1 = 0$$
$$c_2 = 0.$$

Hence, σ is regular.

21 OCT 22

$U \subseteq \mathbb{R}^n$ (open) , $f : U \to \mathbb{R}^n$

$f : U \to \mathbb{R}$

$(f \circ \gamma)'(t)$

$= \cancel{\nabla f(\gamma(t)) \cdot \gamma'(t)}$

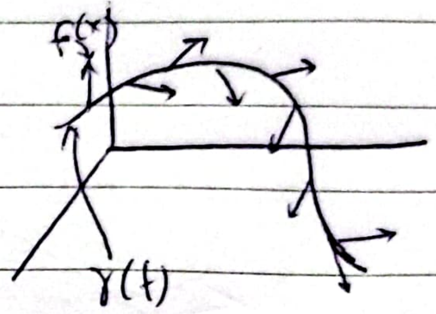
$= (\text{grad} f)(\gamma(t)) \cdot \gamma'(t)$

$(n=3)$

$$f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} f_1\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) \\ f_2(\quad) \\ f_3(\quad) \end{bmatrix}$$

$: U' \to \mathbb{R}^3$

Now : $$(f \circ \gamma)'(t) = \begin{bmatrix} (f_1 \circ \gamma)'(t) \\ (f_2 \circ \gamma)'(t) \\ (f_3 \circ \gamma)'(t) \end{bmatrix}$$



$$= \begin{bmatrix} \nabla f_1 \cdot \gamma' \\ \nabla f_2 \cdot \gamma' \\ \nabla f_2 \cdot \gamma' \end{bmatrix}$$

dot product

$$= \begin{bmatrix} \nabla f_1 \\ \nabla f_2 \\ \nabla f_2 \end{bmatrix} \gamma'$$

row

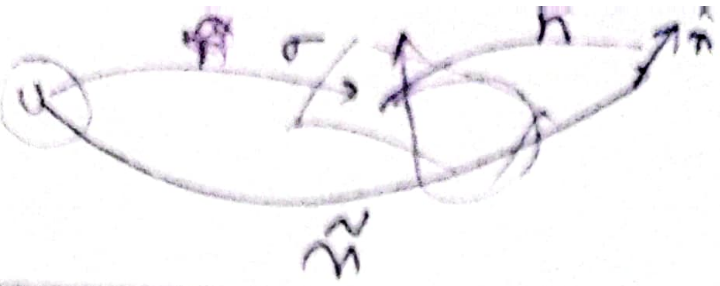
matrix multiplication

$$\underbrace{\begin{bmatrix} \partial_x f_1 & \partial_y f_1 & \partial_z f_1 \\ \partial_x f_2 & \partial_y f_2 & \partial_z f_2 \\ \partial_x f_3 & \partial_y f_3 & \partial_z f_3 \end{bmatrix}}_{\text{Jacobian of } f} \begin{bmatrix} \vec{X} \end{bmatrix}$$

Matrix del at $p \in U$.

$(J_p f)$

Jacobian of f $\quad (J_p f)(x) : U \to M_3(\mathbb{R})$



$$\nabla_i(\nabla E) = \nabla^2 E$$

Hence, $(f \circ \gamma^r)'(t) = \left(\underset{\text{where } p=\gamma(t)}{(J_p f)} \right) \left[\gamma'(t) \right]$

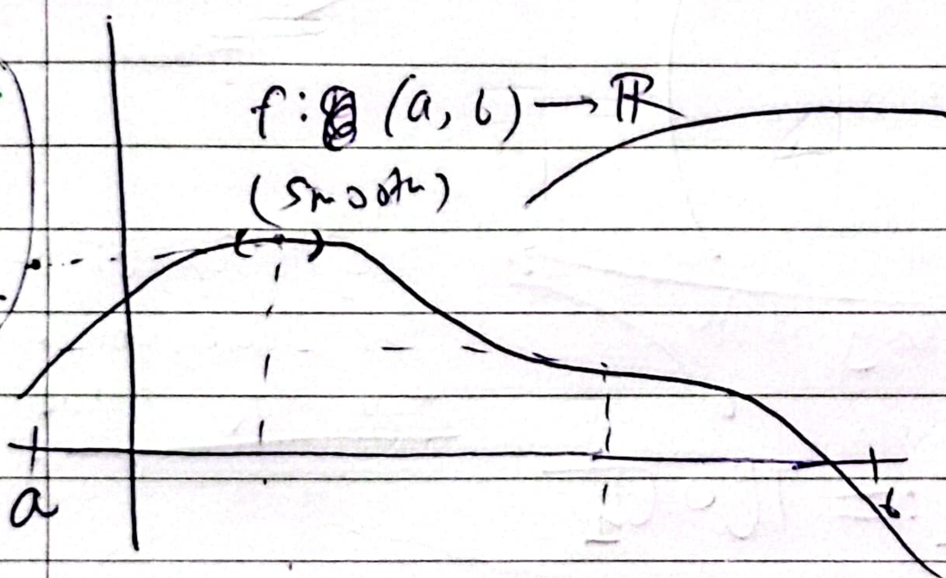
Note:

(i) $(J_p f)(\hat{X}) = \dfrac{\partial f}{\partial \hat{X}}(p)$

(ii) $f: \underset{\subseteq \mathbb{R}}{I} \longrightarrow \mathbb{R}$, $(J_p f)(x)$ is just $\left[\dfrac{df}{dx}(x)\right]$

Inverse fxn thm for $\mathbb{R}$

$f: (a,b) \to \mathbb{R}$ (smooth)



(Inverse fxn thm)

claim:

If $f'(a) \neq 0$ at

$\exists$ a interval I

$a \in I$, s.t.

$\exists f^{-1}: (\,) \longrightarrow I$

which is smooth.

generalize to

$f: \underset{\subseteq \mathbb{R}^3}{U} \to \mathbb{R}^3$

Inverse fxn thm for $\mathbb{R}^3$

(Inverse fxn thm)

Claim:

$\det(J_p f) \neq 0$

at some p.

Then $\exists$ a $V \subseteq U$

s.t. $p \in V$, s.t. $f(V)$ is open.

$\exists f^{-1}: f(V) \to V$

"graph"

this should be a LT that preserves volume?

Say $\sigma_1 : U_1 \longrightarrow S \subseteq \mathbb{R}^3$

$\sigma_2 : U_2 \nearrow$ both smooth & regular

$\Phi := \sigma_2^{-1} \circ \sigma$

Claim: Φ is a diffeom.

$$\sigma_1(x,y) = \begin{bmatrix} \sigma_1^1(x,y) \\ \sigma_1^2(x,y) \\ \sigma_1^3(x,y) \end{bmatrix}$$

proof:

① $\partial_x \sigma_1 \times \partial_y \sigma_1 \neq \vec{0}$ (regular) $\Rightarrow$

② $\begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ & \partial_x\sigma_1 & \\ & \partial_y\sigma_2 & \end{pmatrix}$ there is a component $\neq 0$

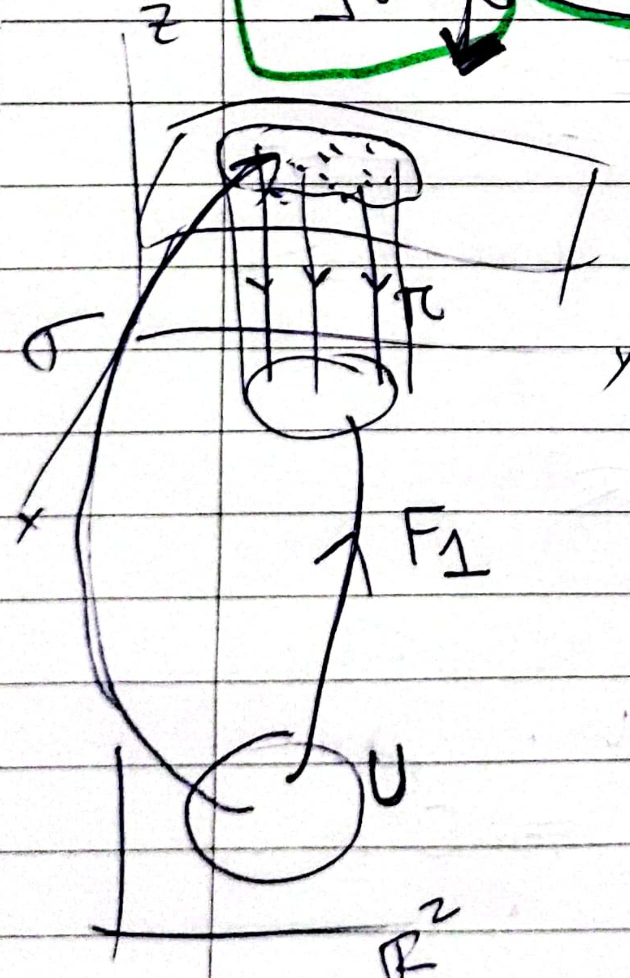
③ choose the part

$$\begin{pmatrix} \partial_x\sigma_1^1 & \partial_y\sigma_2^1 \\ \partial_x\sigma_1^2 & \partial_y\sigma_2^2 \\ \partial_x\sigma_1^3 & \partial_y\sigma_2^3 \end{pmatrix}$$

$\exists$ a π

say it is

④ projection of the surface on the x-y plane



⑤ $F_1 := \pi \circ \sigma_1$

and $(J_p F_1) = \begin{pmatrix} \partial_x\sigma_1^1 & \partial_y\sigma_2^1 \\ \partial_x\sigma_1^2 & \partial_y\sigma_2^2 \end{pmatrix}$

$\det(J_p F_1) = \underbrace{\times}_{z \text{ component of } \partial_x\sigma_1 \times \partial_y\sigma_1 \neq 0} \neq 0$

$\neq 0$!

bijective!

because π, σ_1 are bij.

(6) same here → $F_2 := \pi \circ \sigma_2 \longrightarrow$ bij.!

then $\Phi = \sigma_2^{-1} \circ \sigma_1 = \underbrace{\sigma_2^{-1} \circ \pi^{-1}} \circ \pi \circ \sigma_1$

$$= \underbrace{F_2^{-1} \circ F_1}$$

inverse fxn thm

so nice! $F_1 : U_1 \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

smooth $F_2 : U_2 \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

(7) $\exists F_2^{-1}$, and smooth

~~And now~~ ⇒ ~~$(F_2^{-1} \circ F_1)$ is smooth~~

~~and det( ) $\neq 0$~~

then $\underbrace{F_2^{-1}}_{\text{smooth}} \circ \underbrace{F_1}_{\text{smooth}} = \Phi \Rightarrow \underbrace{\Phi}_{\text{smooth}}$

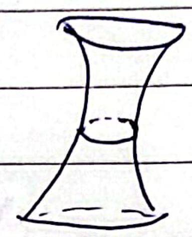
Similarly, Φ^{-1} exists and is smooth!

Hence, proved.

$\boxed{31OCT22}$

Sphere: $S^2 := \{ x \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1 \}$

Hyperboloid: $\{ x \in \mathbb{R}^3 : x_1^2 + x_2^2 - x_3^2 = 1 \}$



Plane: $\{ x \in \mathbb{R}^3 : x_1 + x_2 = 0 \}$

or U?

* Given a $f(x) : \mathbb{R}^3 \to \mathbb{R}$, smooth

$S_f := \{ x \in \mathbb{R}^3 : f(x) = 0 \}$.

Under what conditions, is S_f a surface?

(counterexample)

$f(x) := x_1^2 + x_2^2 + x_3^2$

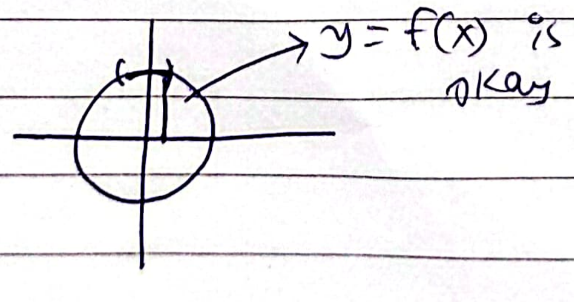
S_f is one ~~point~~ point $\to \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

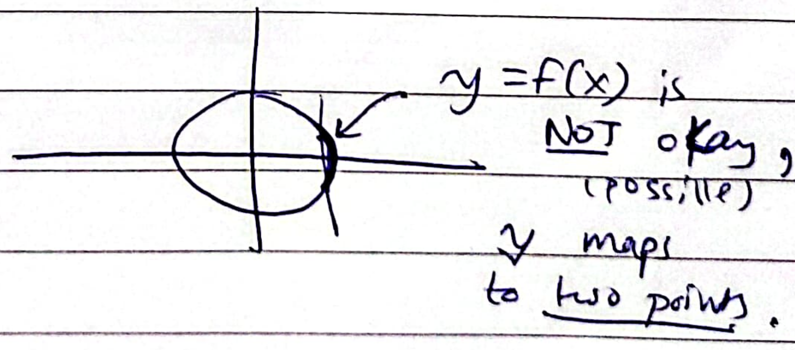
which is not a surface.

$\to$ So we NEED some conditions!

* HOW can we go $f(x) = 0 \rightsquigarrow$ "$z = g(x, y)$" ?

(1)





(2) $x^4 + 5x^3y + 7x^2y^2 \cancel{z} + y^9$

Cannot easily have $y = f(x)$:(.

(3) $\Longrightarrow$ We do not need to have a nice $z = g(x,y)$, $\rightarrow$ we just need to Know there exists one.

* locally ONLY, on a open neighborhood of $x \in \mathbb{R}^3$.

$\boxed{\text{Target: find a } \sigma \text{ for } S_f}$

or multiple $\Rightarrow$ just locally at $x \in \mathbb{R}^3$.

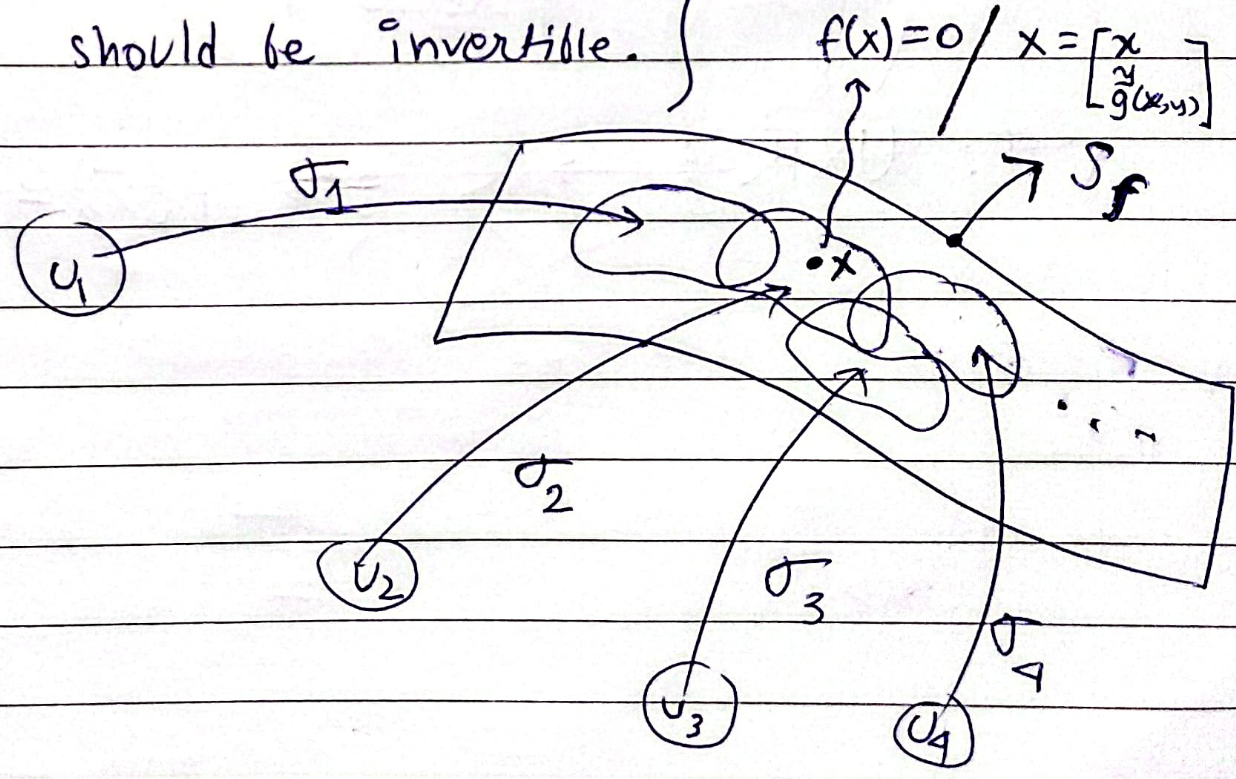
To find $g(x,y)$ s.t. $\sigma : U \rightarrow \mathbb{R}^3$

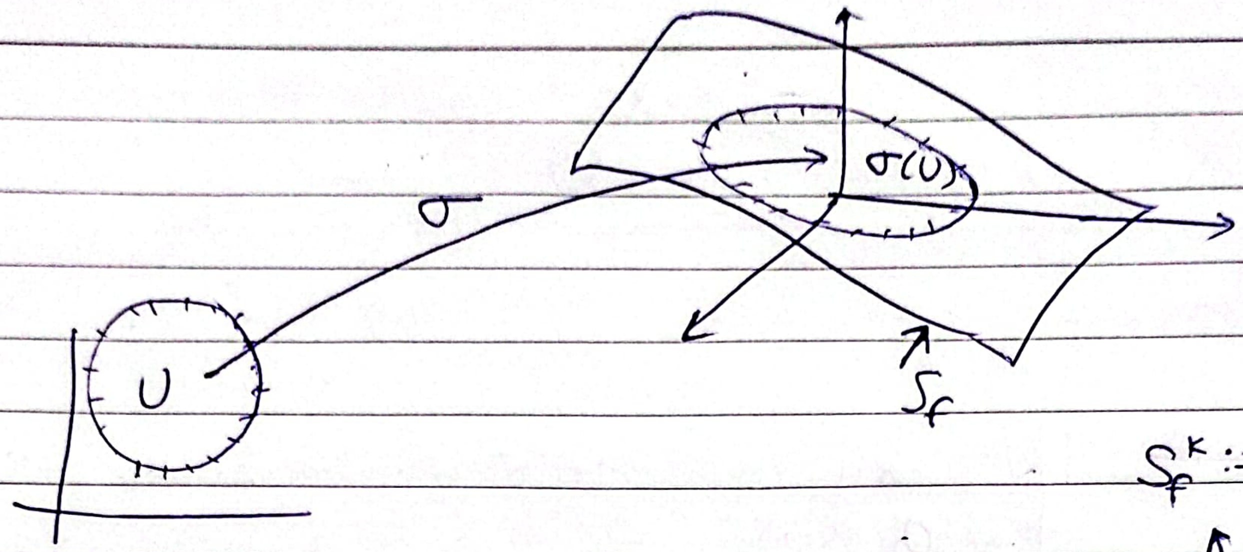
$\sigma(x,y) := (x, y, g(x,y))$

Such that $\sigma_i(U_i) \subseteq S_f$

and $\{\sigma_i\}$ covers S_f $\boxed{\bigcup_i \sigma_i(U_i) = S_f}$

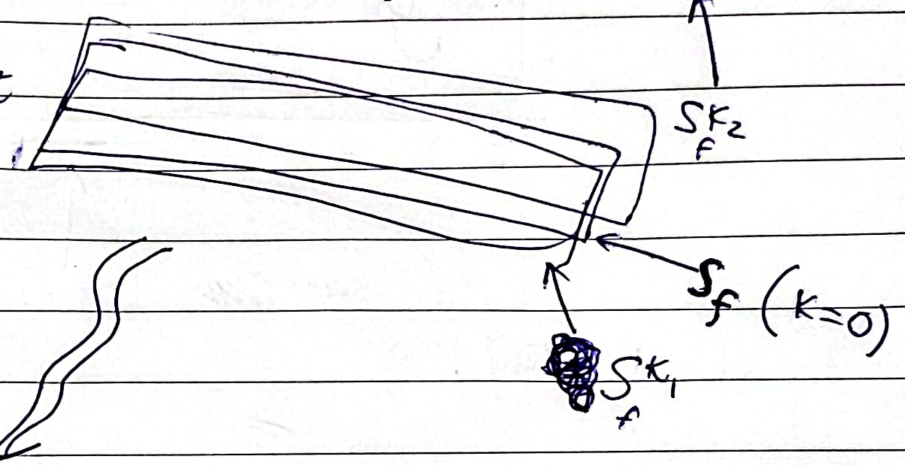
$\rightarrow$ { g should be smooth. g should be invertible. }

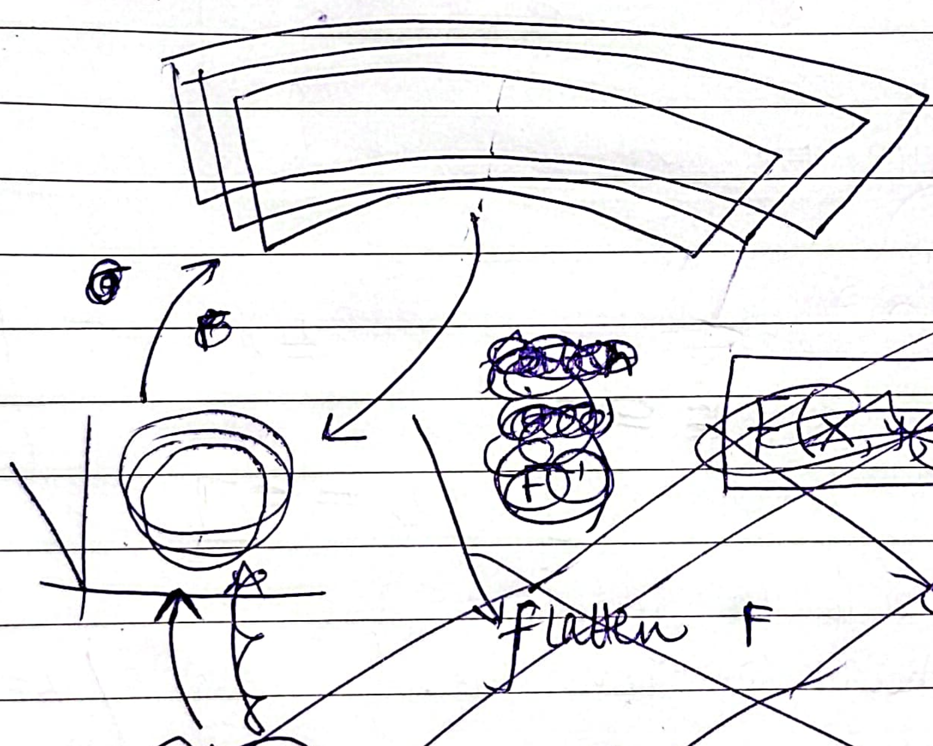




$S_f^k := \{\vec{x}\in\mathbb{R}^3 : f(\vec{x})=k\}$

Look at

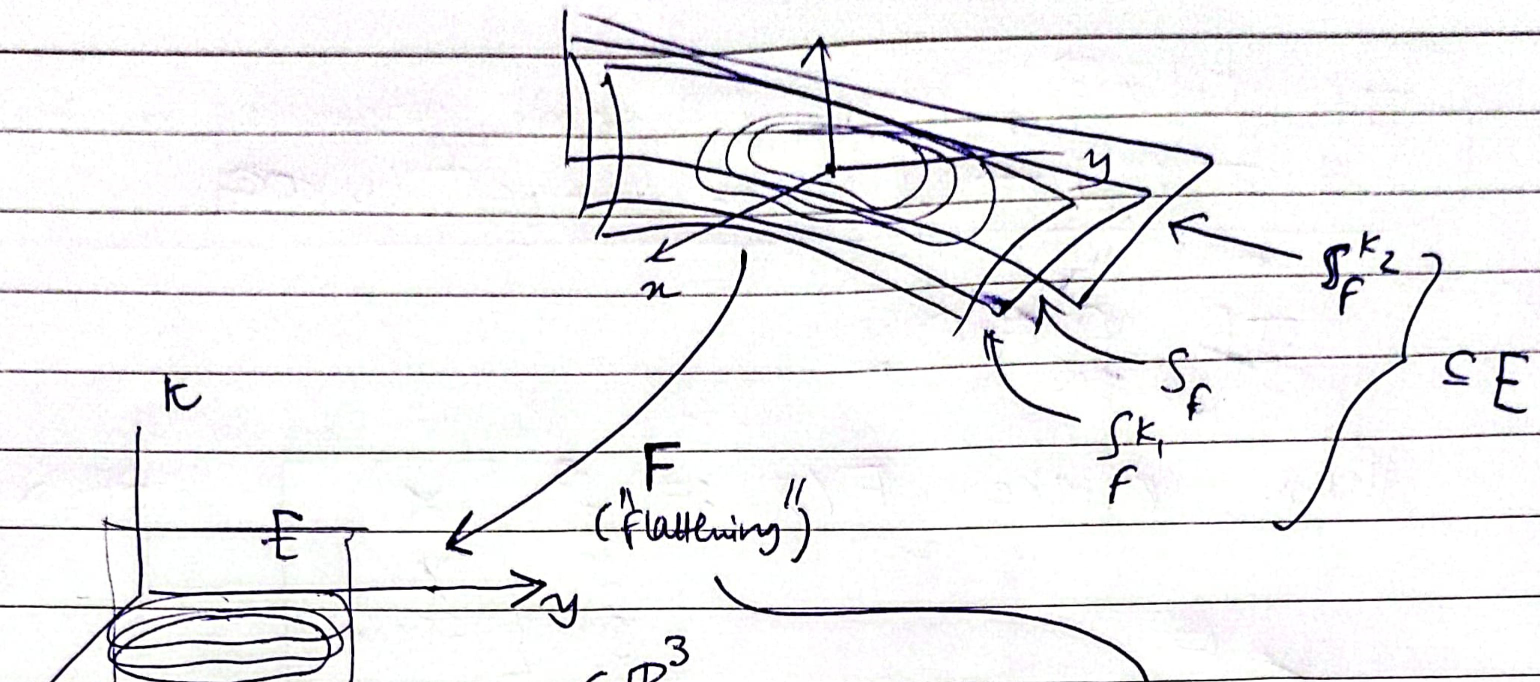




Flatten F

$\cancel{F: U\times\mathbb{R} \to \mathbb{R}^3 \quad F(\vec{x}) = \begin{bmatrix} x \\ y \\ f(x,y,z)\end{bmatrix} = \begin{bmatrix} x \\ y \\ k\end{bmatrix}}$

$$F(x): E^{\subseteq \mathbb{R}^3} \longrightarrow \mathbb{R}$$



$$F: E^{\subseteq \mathbb{R}^3} \longrightarrow \mathbb{R}^3$$

$$F(\vec{x}) := \begin{bmatrix} x \\ y \\ f(x,y) \end{bmatrix} \begin{array}{l} \left.\begin{array}{l} \\ \\ \end{array}\right\} \text{x-y projection} \\ \longleftarrow \text{value of } k \end{array}$$

If F^{-1} exists, then $F^{-1} : \underbrace{(\ \ )}_{\substack{\text{restricted} \\ z=0}} \longrightarrow S_f$

would be

$F^{-1} = \sigma$!

Now, $(J_p f) = \begin{pmatrix} 1 & 0 & f_x \\ 0 & 1 & f_y \\ 0 & 0 & f_z \end{pmatrix}$

$\Rightarrow \det(J_p f) = f_z$

$\Rightarrow$ So, if $f_z \neq 0$, F^{-1} exists.

* If $f_z = 0$, we can take f_x, f_y if they are $\neq 0$, by taking yz, zx projections while flattening

$\Rightarrow$ If $\nabla f\big|_p \neq \vec{0}$, then we can gurantee $\exists F^{-1}$, and $\exists$ a σ ~~around~~ around p.

(Claim) $\boxed{\text{If } \nabla f|_p \neq \vec{0}, \text{ then we can gurantee a surface patch } \exists\ \sigma : U \to S_f \text{ such that } p \in \sigma(U)}$

* If $f_\alpha \neq 0$,

then take $F(\vec{x}) := \begin{bmatrix} x \\ y \\ f(x,y) \end{bmatrix}$ if $\alpha = 3$

and $\det(J_p F) = f_\alpha \neq 0$.

$\Rightarrow$ then F has a smooth inverse around a small neigh. of p, say it be:

$$\underbrace{G}_{G = F^{-1}}(x, y, \underbrace{z}_{k}) = (x, y, g(x,y,\underbrace{z}_{k}))$$

$(F \circ G) = (x, y, \underbrace{z}_{k}) \Rightarrow (f \circ g) = \underbrace{z}_{k}$

$(G \circ F) = (x, y, \underbrace{f(x,y,z)}_{k}) \Rightarrow (g \circ f) = \underbrace{f(x,y,z)}_{k}$

Then $\underbrace{\sigma(x,y) := G(x,y,\underbrace{0}_{k=0})}_{\vec{x} =}$

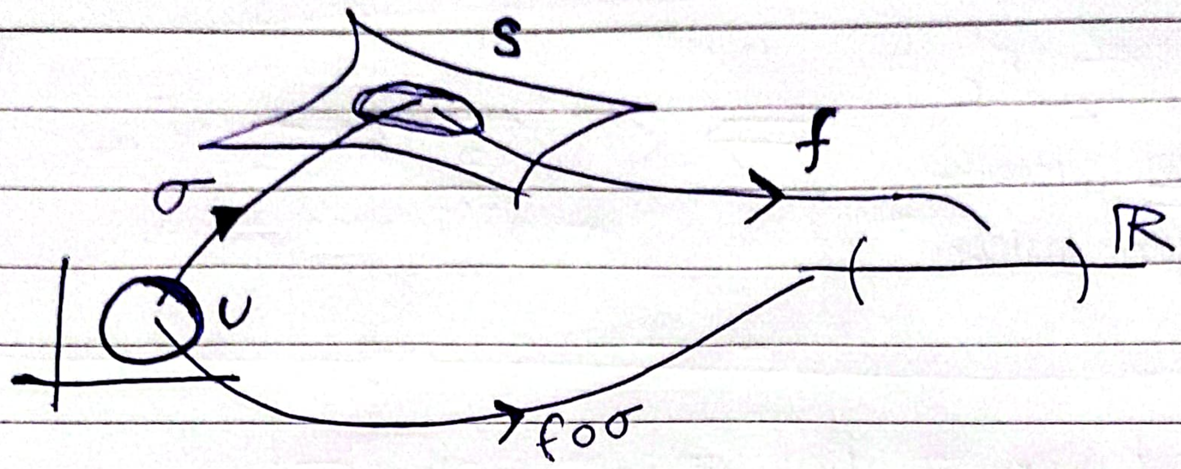
but does $(x, y, g(x,y,0))$ satisfy $f(\vec{x}) = 0$?
or If $\vec{x} \in S_f$

Yes! $F^{-1} = G$

~~$G(x, y, g(x,y,0) = F^{-1}($~~

$\Rightarrow \quad G(x,y,0) = F^{-1}(x,y,0)$

$\Rightarrow \quad G(x,y,0) = \underbrace{(x, y, g(x,y,\underbrace{0}_{k=0}))}_{\text{so this } \in S_f.}$



$f : S \to \mathbb{R}$ then $f\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is not differentiable in conventional definition.

$\sigma \nearrow$

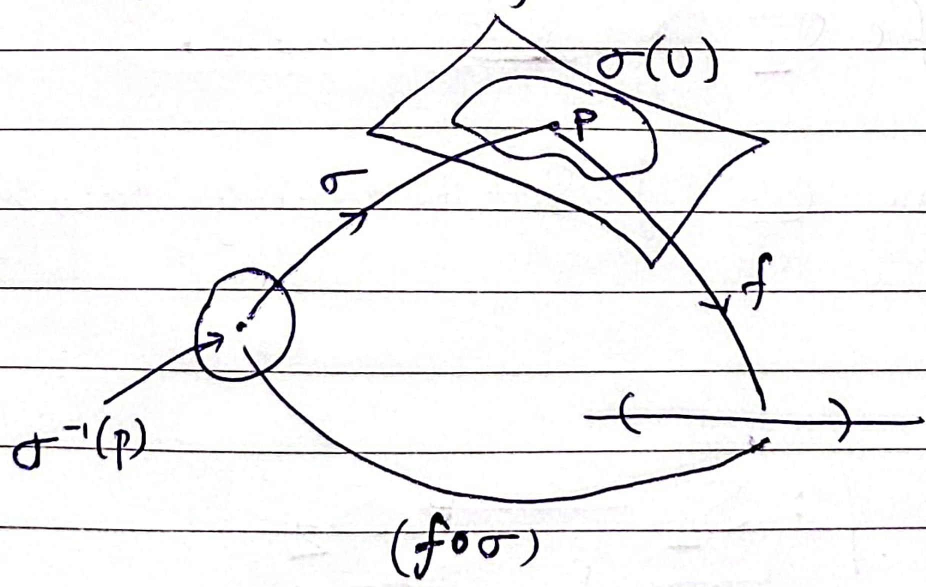
U

$\in S$

$(f \circ \sigma)\begin{pmatrix} x \\ y \end{pmatrix}$

$\in U$

(Definition) $f : S \to \mathbb{R}$ is smooth at $p \in S$ if given a surface patch $\sigma : U \to \sigma(U) \subseteq S$, s.t. $p \in \sigma(U)$, and $(f \circ \sigma)$ is smooth at $\sigma^{-1}(p)$.



? But does this defn work if we change σ?
(is the defn well-defined)

$$\boxed{\text{(Thm) } \sigma_1, \sigma_2 \text{ be regular smooth surface patches} \implies \sigma_2^{-1} \circ \sigma_1 \text{ is a diffeomorphism}}$$

We proved this. Let $\Phi := \sigma_2^{-1} \circ \sigma_1$

Then, $\sigma_1 = \sigma_2 \circ \Phi$

"change of coordinates"

Now, given a $f : S \to \mathbb{R}$ which is smooth (by new dfn) with a patch σ_2,

$$f \circ \sigma_1 = \underbrace{f \circ \sigma_2}_{} \circ \Phi$$

If $f \circ \sigma_2$ was smooth,

then $\underbrace{(f \circ \sigma_2)}_{\text{smooth}} \circ \underbrace{\Phi}_{\text{smooth}}$ is also smooth (composition of smooth fxns are smooth).

$\Rightarrow f \circ \sigma_1$ is also smooth.

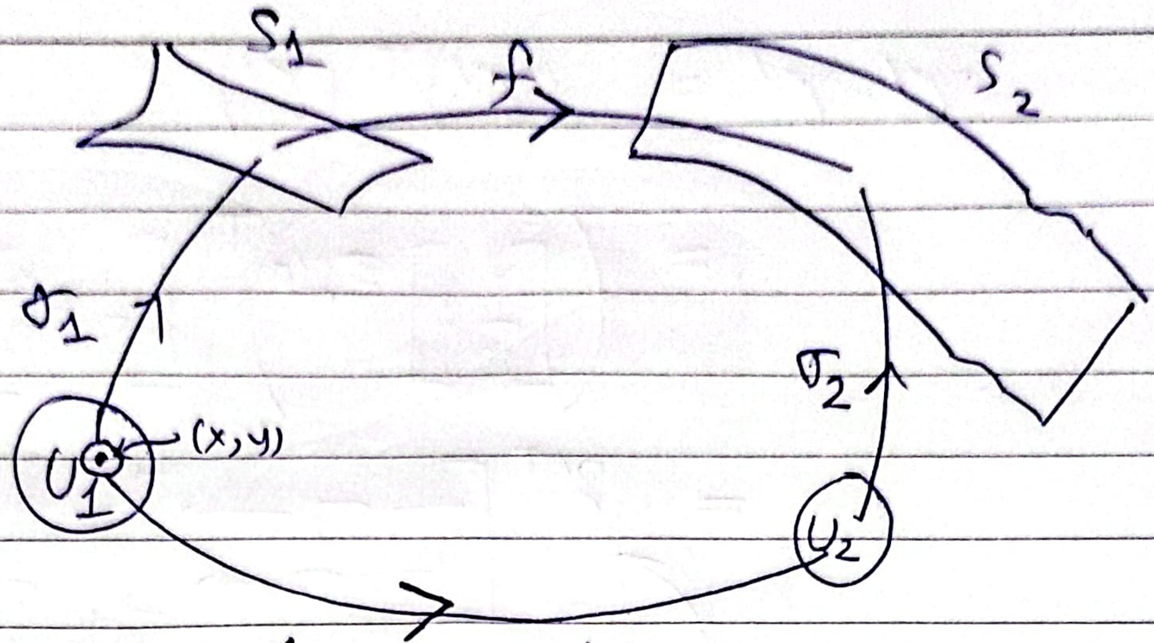
So, definition of $f : S \to \mathbb{R}$ being smooth is well-defined.

Say now,

$$f : \underbrace{S_1}_{} \longrightarrow \underbrace{S_2}_{}$$

both surfaces in $\mathbb{R}^3$.

How can we define this function being smooth?



Take the fxn $(\sigma_2^{-1} \circ f \circ \sigma_1)(x,y)$ and that is a function

$$\sigma_2^{-1} \circ f \circ \sigma_1 : \underset{(\text{open})}{U_1^{\subseteq \mathbb{R}^2}} \longrightarrow \underset{(\text{open})}{U_2^{\subseteq \mathbb{R}^2}}$$

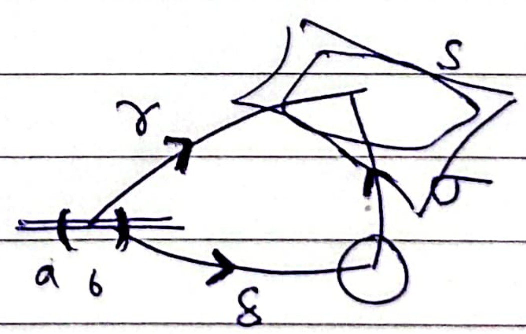
(Defn) ~~If~~ Given $f : S_1 \longrightarrow S_2$, if $\sigma_2^{-1} \circ f \circ \sigma_1$ is smooth then we say f is smooth.

<u>Exercise:</u> change σ_1, σ_2 and check if f remains "smooth" in new defn.

—— x ——

Say, $\gamma : (a,b) \longrightarrow \underset{(\text{surface})}{S^{\subseteq \mathbb{R}^3}}$

$$\int_{[c,d]} \|\gamma'\| \longleftarrow \text{arc-length of the curve in } \mathbb{R}^3$$



$$\gamma = \sigma \circ \delta$$

$$\Rightarrow \quad \gamma' = [\sigma_x \ \ \sigma_y] \, \delta'$$

evaluated at $\delta(t)$

$$\begin{bmatrix} \uparrow & \uparrow \\ \sigma_x & \sigma_y \\ \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} \delta_x' \\ \delta_x' \end{bmatrix}$$

$$\gamma' \cdot \gamma' = (\gamma')^T (\gamma')$$

$$= \left(\begin{bmatrix} \uparrow & \uparrow \\ \sigma_x & \sigma_y \\ \downarrow & \downarrow \end{bmatrix} \delta' \right)^T \begin{bmatrix} \uparrow & \uparrow \\ \sigma_x & \sigma_y \\ \downarrow & \downarrow \end{bmatrix} \delta'$$

$$= \underbrace{\delta'^T}_{[\delta'_x \ \delta'_y]} \underbrace{\begin{bmatrix} \leftarrow \sigma_x \rightarrow \\ \leftarrow \sigma_x \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow & \uparrow \\ \sigma_x & \sigma_y \\ \downarrow & \downarrow \end{bmatrix}} \underbrace{\delta'}_{\begin{bmatrix} \delta'_x \\ \delta'_y \end{bmatrix}}$$

$$= \delta'^T \underbrace{\begin{bmatrix} \sigma_x \cdot \sigma_x & \sigma_x \cdot \sigma_y \\ \sigma_x \cdot \sigma_y & \sigma_y \cdot \sigma_y \end{bmatrix}}_{\text{depends only on } \sigma, \text{ not on curve } \gamma} \delta'$$

Symmetric matrix (obviously)

Thus,

$$\int \|\gamma'\| = \int \sqrt{\delta'^T \underbrace{\begin{bmatrix} \sigma_x \cdot \sigma_x & \sigma_x \cdot \sigma_y \\ \sigma_x \cdot \sigma_y & \sigma_y \cdot \sigma_y \end{bmatrix}}_{\text{reflects the geometry of the surface}} \delta'}$$

the matrix is called the first fundamental form.

Next

$$\text{area} = \int_U \underbrace{\sqrt{\det\begin{bmatrix} \sigma_x\cdot\sigma_x & \sigma_x\cdot\sigma_y \\ \sigma_x\cdot\sigma_y & \sigma_y\cdot\sigma_y \end{bmatrix}}}_{}$$

$$= \|\sigma_x \times \sigma_y\|$$

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9 NOV 22

$$\begin{tikzcd}
S_1 \arrow[r, "f"] & S_2 \\
U_1 \arrow[u, "\sigma_1"] \arrow[r, dashed] & U_2 \arrow[u, "\sigma_2"']
\end{tikzcd}$$

- $\sigma_2^{-1} \circ f \circ \sigma_1$ is smooth $\overset{\text{defn}}{\Longrightarrow}$ f is smooth.

* Take $\gamma : (a, b) \longrightarrow S_1$
and $\gamma(t_0) = p$

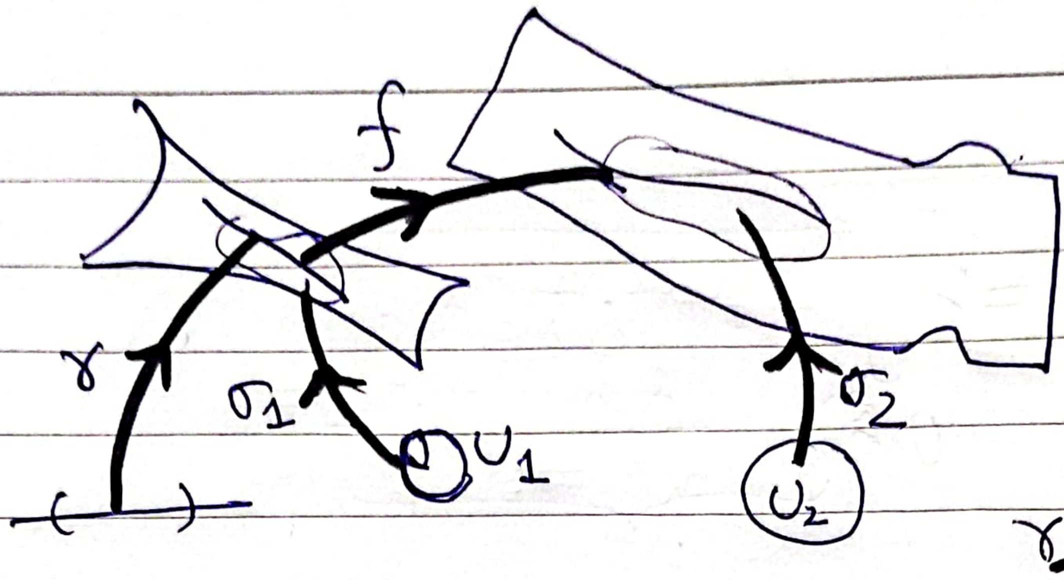
Say $v := \gamma'(t_0)$
$\Rightarrow$ $v \in T_p S_1$ (by definition)

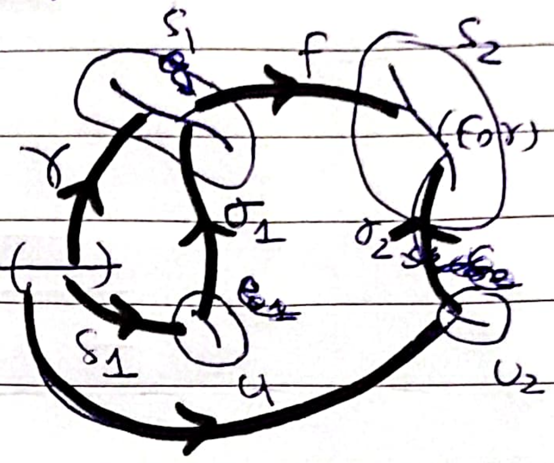
* Consider f (a smooth $S_1 \xrightarrow{f} S_2$)

then $f(\gamma(t))$ is a curve on S_2

→ What is $(f\circ\gamma)'(t_0)$?

Say $\vec{w} := (f\circ\gamma)'(t_0)$, $\vec{w} \in T_{\not{p}} S_2$





$\rightsquigarrow \quad \delta_1 := \sigma_1^{-1} \circ (\gamma)$

* $f \circ \gamma = f \circ \sigma_1 \circ \delta_1$

$\delta_2 := \sigma_2^{-1} \circ (f \circ \gamma)$

$$\boxed{\gamma' = \sigma_{1x}\,\delta'_{1x} + \sigma_{1y}\,\delta'_{1y} \qquad (f\circ\gamma)' = \sigma_{2x}\,\delta'_{2x} + \delta_{2y}\,\delta'_{2y}}$$

$$\delta_2 = \underbrace{\sigma_2^{-1} \circ f \circ \sigma_1}_{\substack{\text{some smooth} \\ g:\,U_1 \to U_2}} \circ \underbrace{\delta_1}_{\text{smooth}}$$

$\Rightarrow \quad \delta_2 = g \circ \delta_1 \quad \}$ smooth

$\Rightarrow \quad \delta_2' = g_x\,\delta'_{1x} + g_y\,\delta'_{1y}$

$$\Rightarrow \quad \boxed{\delta_2'(t) = \underbrace{\begin{bmatrix} \uparrow & \uparrow \\ g_x & g_y \\ \downarrow & \downarrow \end{bmatrix}}_{\substack{\text{Jacobian of} \\ g \text{ at } \delta_1(t) \\ (2\times 2 \text{ matrix})}} \underbrace{\begin{bmatrix} \delta'_{1x} \\ \delta'_{1y} \end{bmatrix}}_{\delta_1'(t)}}$$

$$\delta_2'\Big|_t = \underbrace{\left(J_{\delta_1(t)}\, g\right)\left(\delta_1'(t)\right)}_{}$$

only depends on $\delta_1(t)$, $\delta_1'(t)$

Now, $v = \gamma'(t_0) = \begin{bmatrix} \sigma_{1x} & \sigma_{1y} \end{bmatrix} \begin{bmatrix} \delta_1'(t_0) \end{bmatrix}$

$$W = (f\circ\gamma)'(t_0) = \begin{bmatrix} \sigma_{2x} & \sigma_{2y} \end{bmatrix} \begin{bmatrix} \delta_2'(t_0) \end{bmatrix}$$

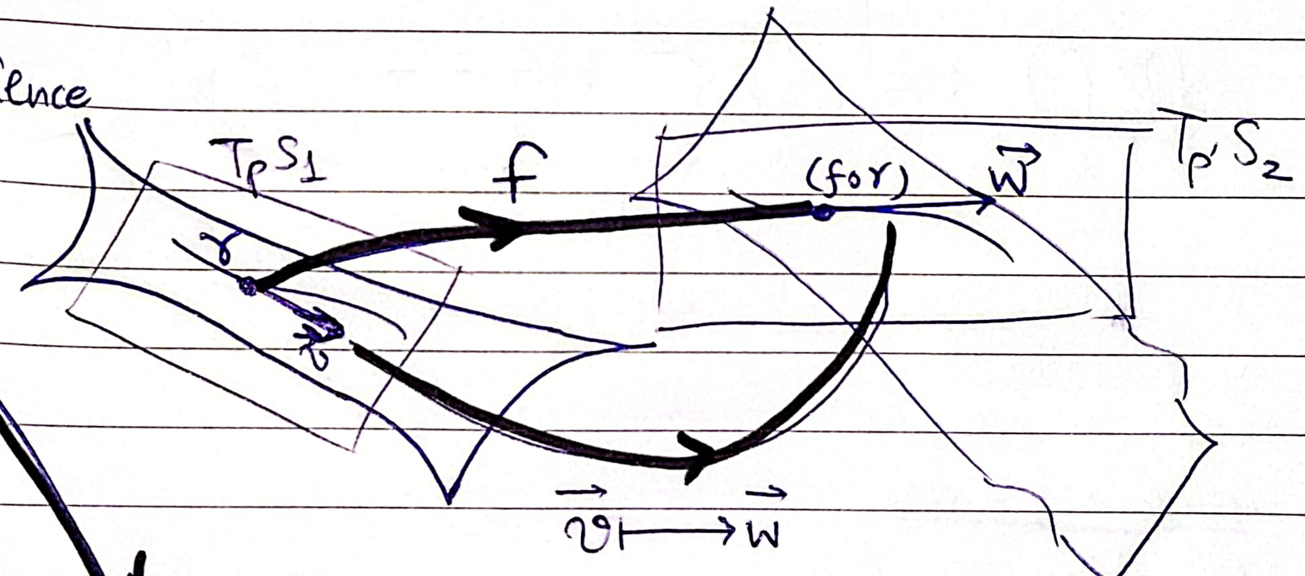
$$\vec{W} = \begin{bmatrix} \sigma_{2x} & \sigma_{2y} \end{bmatrix} \left(J_{\delta_1(t_0)}\, g\right) \delta_1'$$

(left inverse)

$$= \underbrace{\begin{bmatrix} \sigma_{2x} & \sigma_{2y} \end{bmatrix} \left(J_{\delta_1(t_0)}\, g\right) \begin{bmatrix} \sigma_{1x} & \sigma_{2y} \end{bmatrix}^{-1}}_{} \vec{V}$$

Only depends on the point p, $f(p)$ etc.

Hence



converts velocities on S_1 to one in S_2!

conclusion: $(f\circ\gamma)'(t_0)$ depends on $\gamma'(t_0)$, $\gamma(t_0)$

Therefore, we can define

Linear transformation

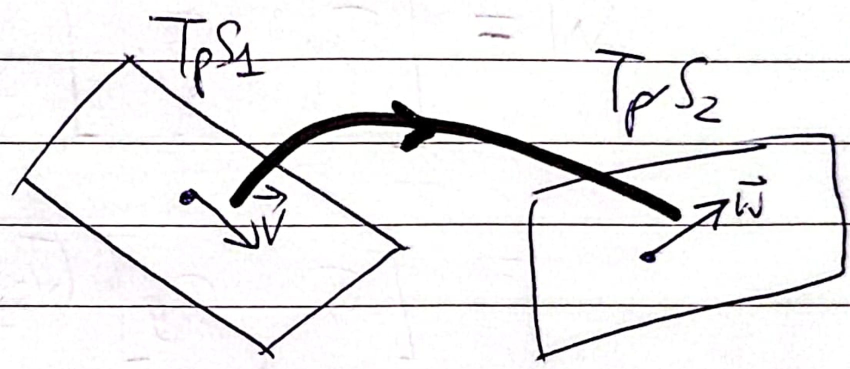
$$(D_p f) : T_pS_1 \xrightarrow{\sim} T_{p'}S_2$$

as $$\Phi(D_p f) := \underbrace{(D_p \sigma_1)\circ(D_{p'} g)\circ(D_{p''}\sigma_2)^{-1}}$$

matrix form $\left\{ \begin{bmatrix}\sigma_{1x} & \sigma_{1y}\end{bmatrix}(J_{p'}g)\underbrace{\begin{bmatrix}\sigma_{2x} & \sigma_{2y}\end{bmatrix}^{-1}}_{\text{left inverse (of matrix)}} \right.$

such that

$$(D_p f)(\vec{v}) = \vec{w}$$



Here,

$$(D_p f) := \underbrace{(D_p \sigma_1)}\circ\underbrace{(D_{p'} g)}\circ\underbrace{(D_p \sigma_2)^{-1}}$$

total derivatives in conventional definition

non differentiable in conventional defn; so we "define" a new way AND define the total derivative

12 NOV 22

* Given a smooth $f : S_1 \longrightarrow S_2$, we define a map $(D_p f) : T_p S_1 \xrightarrow{\sim} T_{f(p)} S_2$, s.t.

$$(D_p f)(\underbrace{\dot{\gamma}}) = (\underbrace{f \circ \gamma})'$$

converts velocities in $T_p S_1$ to velocities in $T_{f(p)} S_2$

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* $$K_n := \gamma'' \cdot \hat{n} = -\gamma' \cdot (\hat{n} \circ \gamma)'$$

$$= + \gamma' \cdot \mathcal{W}_p(\gamma')$$

$$\mathcal{W} := -(D_p \tilde{n}) \circ (D_{p'} \sigma)^{-1}$$

$p' = \sigma^{-1}(p)$

curve property

surface property

$$\mathcal{W}_p(v) := -(\hat{n} \circ \gamma)'$$
$v \in T_p S$

$$(\hat{n} \circ \gamma) \cdot (\hat{n} \circ \gamma) = 1$$

$$\Rightarrow \underbrace{(n \circ \gamma)'} \cdot (n \circ \gamma) = 0$$

$$\Rightarrow \mathcal{W}_p(v) \cdot \hat{n}(p) = 0, \quad \forall\, p \in S, \ v \in T_p S$$

$\therefore \mathcal{W}_p : T_p S \longrightarrow T_p S$.

Knowing where the basis map to is enough.

$$\Rightarrow \mathcal{W}_p(v) = c_1 \mathcal{W}_p(\sigma_x) + c_2 \mathcal{W}_p(\sigma_y)$$

$$\mathcal{W}_p(\sigma_x) = a\,\sigma_x + b\,\sigma_y$$
$$\mathcal{W}_p(\sigma_y) = c\,\sigma_x + d\,\sigma_y$$

$$\gamma_{\sigma_x} := \sigma(t, y_0)$$

$$w_p(\sigma_x) = -(\hat{n} \circ \gamma_{\sigma_x})' = -(\hat{n} \circ \sigma(t, y_0))'$$

$$= -\left(\tilde{n} \circ \begin{bmatrix} t \\ y_0 \end{bmatrix}\right)'$$

$$= -\tilde{n}_x$$

In other way,

$$w_p \xi := -(D_p \tilde{n}) \circ (D_p \sigma)^{-1}$$

$$w_p(\sigma_x) = -(D_p \tilde{n}) \circ \underbrace{(D_p \sigma^{-1}) \sigma_x}$$

$$= -\begin{bmatrix} n_x & n_y \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= -n_x$$

Similarly $w_p(\sigma_y) = -\tilde{n}_y$

$$\begin{cases} w_p(\sigma_x) = -\tilde{n}_x = a\sigma_x + b\sigma_y \\ w_p(\sigma_y) = -\tilde{n}_y = c\sigma_x + d\sigma_y \end{cases}$$

$$-\tilde{n}_x \cdot \sigma_x = a\sigma_x^2 + b\sigma_x \cdot \sigma_y$$

$$-\tilde{n}_x \cdot \sigma_y = a\,\sigma_x \cdot \sigma_y + b\,\sigma_y^2$$

$$-\tilde{n}_y \cdot \sigma_x = c\,\sigma_x^2 + d\,\sigma_x \cdot \sigma_y$$

$$-\tilde{n}_y \cdot \sigma_y = c\,\sigma_x \cdot \sigma_y + d\,\sigma_y^2$$

$$(\tilde{n} \cdot \sigma_x) = 0$$

$$\Rightarrow \tilde{n}_x \cdot \sigma_x + \underbrace{\tilde{n} \cdot \sigma_{xx}}_{L} = 0$$

$$\begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \sigma_x^2 & \sigma_x \cdot \sigma_y \\ \sigma_x \cdot \sigma_y & \sigma_y^2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} \sigma_x^2 & \sigma_x \sigma_y \\ \sigma_x \cdot \sigma_y & \sigma_y^2 \end{pmatrix}^{-1}$$

In other way; $\quad -(\tilde{n}\circ\delta)' = -(D_p\tilde{n})\underbrace{(\delta')}_{\text{but we need } \dot{\gamma} \text{ as input}}$

$$\mathcal{W} := -(D_p\tilde{n})\circ(D_p\sigma)^{-1}$$

s.t. $\quad (D_p\sigma)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\sigma_x + x_2\sigma_y$

and $\quad (D_p\sigma^{-1})(x_1\sigma_x + x_2\sigma_y) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

So, writing $\underbrace{(x_1\sigma_x + x_2\sigma_y) \in T_pS}_{\text{any } v\in T_pS}$ as $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ in $\{\sigma_x, \sigma_y\}$ basis

$$\mathcal{W}(x_1\sigma_x + x_2\sigma_y) = \underbrace{-(D_p\tilde{n})\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}$$

~~and we obtain~~

$$= +\big(x_1(-n_x) + x_2(-n_y)\big)$$

obviously, $\quad -n_x = \underbrace{a\sigma_x + b\sigma_y}_{[D_p\sigma]\begin{bmatrix} a \\ b \end{bmatrix}} \qquad -n_y = \underbrace{c\sigma_x + d\sigma_y}_{[D_p\sigma]\begin{bmatrix} c \\ d \end{bmatrix}}$

$$\mathcal{W}(x_1\sigma_x + x_2\sigma_y) = +\left(x_1[D_p\sigma]\begin{bmatrix} a \\ b \end{bmatrix} + x_2[D_p\sigma]\begin{bmatrix} c \\ d \end{bmatrix}\right)$$

$$= +[D_p\sigma]\underbrace{\left\{x_1\begin{bmatrix} a \\ b \end{bmatrix} + x_2\begin{bmatrix} c \\ d \end{bmatrix}\right\}}$$

$$= +[D_p\sigma]\cdot\begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

So, $\quad \mathcal{W}(x_1\sigma_x + x_2\sigma_y) = -[D_p\tilde{n}]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{+[D_p\sigma]\begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\text{figure out } \begin{bmatrix} a & c \\ b & d \end{bmatrix}}$

$$[\mathcal{W}_p] = \quad - [\nabla_p \tilde{n}][\nabla_p \sigma]^{-1}$$

in $\mathbb{R}^3$ basis

$\underbrace{\phantom{in\ \mathbb{R}^3\ basis}}_{\text{a } 3\times 3 \text{ matrix}}$

but $[\mathcal{W}_p] \longleftarrow$ in $\{\sigma_x, \sigma_y\}$ basis

$$= \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

a 2×2 matrix

of course!

$$\mathcal{W}_p : \underbrace{T_pS}_{dim=2} \xrightarrow{\sim} \underbrace{T_pS}_{dim=2}$$

$$\begin{bmatrix} \uparrow & \uparrow \\ n_x & n_y \\ \downarrow & \downarrow \end{bmatrix} \qquad \begin{bmatrix} \uparrow & \uparrow \\ \sigma_x & \sigma_y \\ \downarrow & \downarrow \end{bmatrix}$$

notice!

$$-\left[D_p \hat{n}\right] = \left[D_p \sigma\right] \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$\Rightarrow \underbrace{-\left[D_p \sigma\right]^T \left[D_p \hat{n}\right]}_{=:\ \text{SECOND fund. form}} = \underbrace{\left[D_p \sigma\right]^T \left[D_p \sigma\right]}_{=:\ \text{FIRST fund. form}} \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$-\begin{bmatrix} \leftarrow \sigma_x \rightarrow \\ \leftarrow \sigma_y \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow & \uparrow \\ n_x & n_y \\ \downarrow & \downarrow \end{bmatrix} \qquad \begin{bmatrix} \leftarrow \sigma_x \rightarrow \\ \leftarrow \sigma_y \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow & \uparrow \\ \sigma_x & \sigma_y \\ \downarrow & \downarrow \end{bmatrix}$$

$$\Rightarrow \overbrace{\begin{bmatrix} -\sigma_x \cdot n_x & -\sigma_x \cdot n_y \\ -\sigma_y \cdot n_x & -\sigma_y \cdot n_y \end{bmatrix}}^{\text{SECOND}} = \overbrace{\begin{bmatrix} \sigma_x^2 & \sigma_x \cdot \sigma_y \\ \sigma_x \cdot \sigma_y & \sigma_y^2 \end{bmatrix}}^{\text{FIRST}} \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$\Rightarrow \left[\text{FIRST}\right]^{-1} \left[\text{SECOND}\right] = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$\Rightarrow \therefore \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \left[\text{FIRST}\right]^{-1} \underbrace{\left[\text{SECOND}\right]}$$

$$= \begin{bmatrix} \sigma_{xx} \cdot n & \sigma_{xy} \cdot n \\ \sigma_{xy} \cdot n & \sigma_{yy} \cdot n \end{bmatrix}$$

transpose

(both symmetric $A^T = A$)

$(AB)^T = B^T A^T$

$$\therefore \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \left[\text{SECOND}\right] \left[\text{FIRST}\right]^{-1}$$

Notice: $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \dfrac{\det\left[\text{SECOND}\right]}{\det\left[\text{FIRST}\right]}$

A "form" is a function that takes two vector and gives a real number. Like DOT product. Written like "$\langle v_1, v_2 \rangle \in \mathbb{R}$."

FIRST Fund. form

$$\dot{\gamma} \cdot \dot{\gamma}$$

$$= [D_p \sigma\, \delta']^T [D_p \sigma\, \delta']$$

$$= \delta'^T \underbrace{[D_p \sigma]^T [D_p \sigma]}_{\text{first fund. form}} \delta'$$

$\vec{V} \in T_pS \rightsquigarrow V = \lambda_1 \sigma_x \lambda_2 \sigma_y$

Write $\vec{V}$ as $\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$

Say $\vec{V}_1 \Rightarrow \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$, $\vec{V}_2 \Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

~~those~~ and

$$\langle v_1, v_2 \rangle_p^{\text{First}} := v_1^T \begin{pmatrix} \text{First} \\ \text{f.f.} \end{pmatrix} v_2$$

then

$$\left\langle \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}, \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \right\rangle_p^{\text{First}} = V_1 \cdot V_2$$

DOT product in $\mathbb{R}^3$

OR,

$$\underbrace{\langle \delta', \delta' \rangle_p^{\text{First}} = \gamma' \cdot \gamma'}_{\text{first f.f. is about lengths}}$$

SECOND Fund. form

$$k_n = -\dot{\gamma} \cdot (n \circ \delta)'$$

$$= -[D_p \sigma\, \delta']^T [D_p \tilde{n}\, \delta']$$

$$= \delta'^T \underbrace{\left(-[D_p \sigma]^T [D_p \tilde{n}]\right)}_{\text{second fundamental form!}} \delta'$$

$\vec{V} \in T_pS \rightsquigarrow V = \lambda_1 \sigma_x + \lambda_2 \sigma_y$

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$$

(Define)

$$\left\langle \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}, \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \right\rangle_p^{\text{second}} := \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}^T \begin{pmatrix} \text{second} \\ \text{f.f.} \end{pmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

then

$$\underbrace{\langle \delta', \delta' \rangle_p^{\text{second}} = k_n}_{\text{second f.f. is about curvature}}$$

14NOV22

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} := -\underbrace{\begin{pmatrix} \tilde{n}_x \cdot \sigma_x & \tilde{n} \\ \tilde{n} & \tilde{n}_y \cdot \sigma_y \end{pmatrix}}_{\text{second fund form}} \underbrace{\begin{pmatrix} \sigma_x^2 & \sigma_x \cdot \sigma_y \\ \sigma_x \cdot \sigma_y & \sigma_y^2 \end{pmatrix}^{-1}}_{\left(\text{first fund form}\right)^{-1}}$$

2) $$\underbrace{(c_1 \sigma_x + c_2 \sigma_y)}_{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

in σ_x, σ_y basis

$$-\tilde{n}_x \cdot \sigma_x = +\tilde{n} \cdot \sigma_{xx} =: L$$
$$\tilde{n} \cdot \sigma_{xy} =: M$$
$$\tilde{n} \cdot \sigma_{yx} = M$$
$$\tilde{n} \cdot \sigma_{yy} = N$$

Second fund form $= \begin{pmatrix} L & M \\ M & N \end{pmatrix}$

— × —

$$\sigma_{xx} = \underset{1}{\boxed{\Gamma_{11}^1}}\, \sigma_x + \_\_\, \sigma_y + \_\_\, \tilde{n}$$

$$\sigma_{xx} \cdot \hat{n} = \_\_\, \tilde{n} \cdot \tilde{n}$$
$$L = \_\_$$

$$\sigma_{xy} = \_\_\, \sigma_x + \_\_\, \sigma_y + M \tilde{n}$$

$$\sigma_{yy} = \_\_\, \sigma_x + \_\_\, \sigma_y + N \tilde{n}$$

$\Gamma_{ij}^{k} \longleftarrow$ Christoffel symbols

(eq 1)

(i) $\underbrace{\sigma_{xx} \cdot \sigma_x}_{E_x} = \Gamma_{11}^{1} \underbrace{\sigma_x^2}_{E} + \Gamma_{11}^{2} \underbrace{\sigma_x \cdot \sigma_y}_{F} + 0$

$(\sigma_x \cdot \sigma_x)_x$

$= 2\sigma_{xx} \cdot \sigma_x$

$$\Rightarrow \boxed{\frac{E_x}{2} = \Gamma_{11}^{1} E + \Gamma_{11}^{2} F}$$

(ii) $\underbrace{\sigma_{xx} \cdot \sigma_y} = \Gamma_{11}^{1} \underbrace{\sigma_x \cdot \sigma_y}_{F} + \Gamma_{11}^{2} \underbrace{\sigma_y^2}_{G}$

$(\sigma_x \cdot \sigma_y)_x$

$= \sigma_{xx} \cdot \sigma_y + \sigma_x \cdot \sigma_{yx}$

* $\sigma_{xy} \cdot \sigma_x$: $(\sigma_x \cdot \sigma_x)_y$ ~~$= \sigma_{yx} \cdot \sigma_x + \sigma_y \cdot \sigma_{xx}$~~

$= \sigma_{xy} \cdot \sigma_x + \sigma_x \cdot \sigma_{xy}$

$E_{xy} = 2\, \sigma_{xy} \cdot \sigma_x$

$= \frac{E_{xy}}{2}$

Proposition: Γ_{ij}^{k}, $i, j, k \in \{1, 2\}$ only depend on E, F & G and their partial derivatives.

$$\sigma_{xxy} = \frac{\partial}{\partial y}\sigma_{xx} = (\ )\sigma_x + (\ )\sigma_y + (\ )\hat{n}$$

$$\sigma_{xyx} = \frac{\partial}{\partial x}\sigma_{xy} = (\ )\sigma_x + (\ )\sigma_y + (\ )\hat{n}$$

Hence coeffs. are equal.

What we will get is :-

$\Rightarrow$ $LN - M^2$ = in terms of E, F, G and in their partial derivatives (first f.f.)

$$\underbrace{\det\begin{pmatrix} L & M \\ M & N \end{pmatrix}}_{\text{second f.f.}}$$ ⟵ only depends on first f.f.

But $[\mathcal{W}_p]_{\sigma_x,\sigma_y \text{ basis}} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

the determinants are the same

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} L & M \\ M & N \end{pmatrix}\begin{pmatrix} \text{first fund form} \end{pmatrix}^{-1}$$

$\therefore$ $\det(\mathcal{W}_p) = \dfrac{\det\begin{pmatrix} L & M \\ M & N \end{pmatrix}}{\det(\text{first fund form})}$ & only depends on first f.f

$\Rightarrow$ $\det(\mathcal{W}_p)$ ⟵ only depends on first f.f.

Any transformation between surfaces that preserves lengths, will preserve the first fundamental form.

So the determinant of W_p will also be preserved.

But det (W_p) is related to the curvature of the surface at the point.

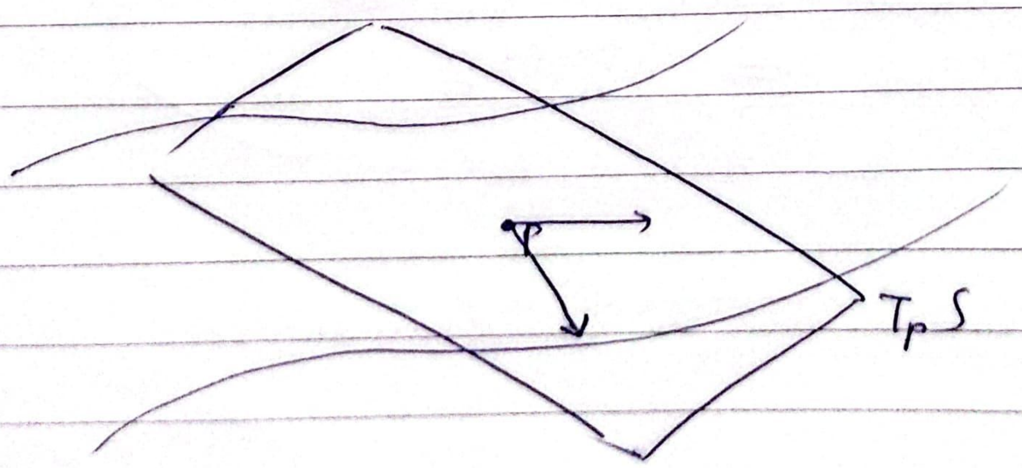
* consider $W_p = \begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}$

and take v_1, v_2 be its eigenvectors; k_1, k_2 be eigenvalues

(i) $W_p(V_i) = k_i V_i$. $\forall\, i \in \{1,2\}$

(ii) if $k_1 \neq k_2$, $\{v_1, v_2\}$ form a unit vector basis,

$[W_p]$ in this basis $\longrightarrow \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$



So, the principal curvatures κ_1, κ_2 might not be same between the surfaces, but det (W_p) = $\kappa_1 \kappa_2$ which definitely gets preserved!

unit

* The W_p map has eigen pairs $\{(\vec{t_1}, k_1), (\vec{t_2}, k_2)\}$

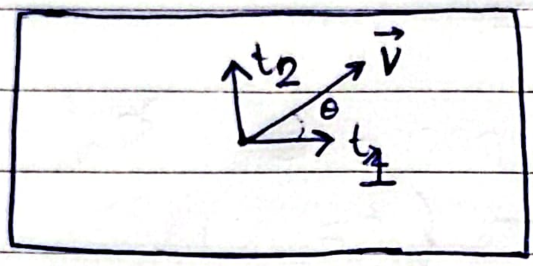
k_n in $\vec{t_1} = t_1 \cdot W_p(t_1) = t_1 k_1 t_1 = k_1$

Similarly for $\vec{t_2}$.

Exercise: W_p is symmetric matrix

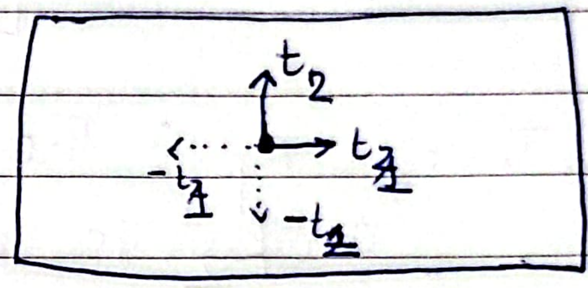
* If $k_1 \neq k_2$, then $t_1 \cdot t_2 = 0$

$\Rightarrow \{t_1, t_2\}$ is an ON-basis for T_pS.

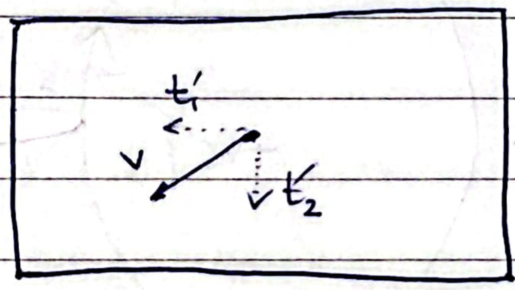


$v \cdot t_2 = \|\vec{v}\| \cos\theta$

$v \cdot t_1 = \|\vec{v}\| \sin\theta$



We have a choice for $\vec{t_1}, \vec{t_2}$.



If $\vec{v}$ was like that but we choose different $\{t_1, t_2\}$.

So that $\forall \vec{V} \in T_pS$

$\vec{V} = \cos\theta\, t_1 + \sin\theta\, t_2$

s.t. $0 \leq \theta \leq \frac{\pi}{2}$

choose $\{t_1, t_2\}$ s.t. this is true.

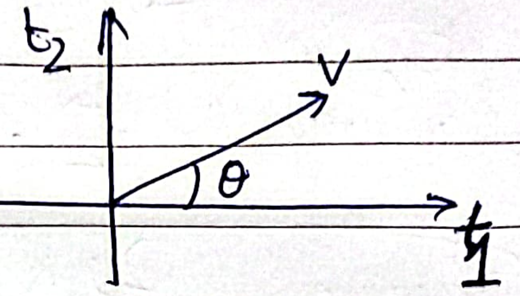
Then $\vec{v} = \cos\theta\, \vec{t_1} + \sin\theta\, \vec{t_2}$, $\cos\theta, \sin\theta \geq 0$.

$W\vec{v} = k_1 \cos\theta\, \vec{t_1} + k_2 \sin\theta\, \vec{t_2}$

$v \cdot W\vec{v} = k_1 \cos^2\theta + k_2 \sin^2\theta$

θ is angle b/w v, t_1.

k_n in direction $\vec{v} = v \cdot 2\mathcal{U}(v) = k_1 \cos^2\theta + k_2 \sin^2\theta$



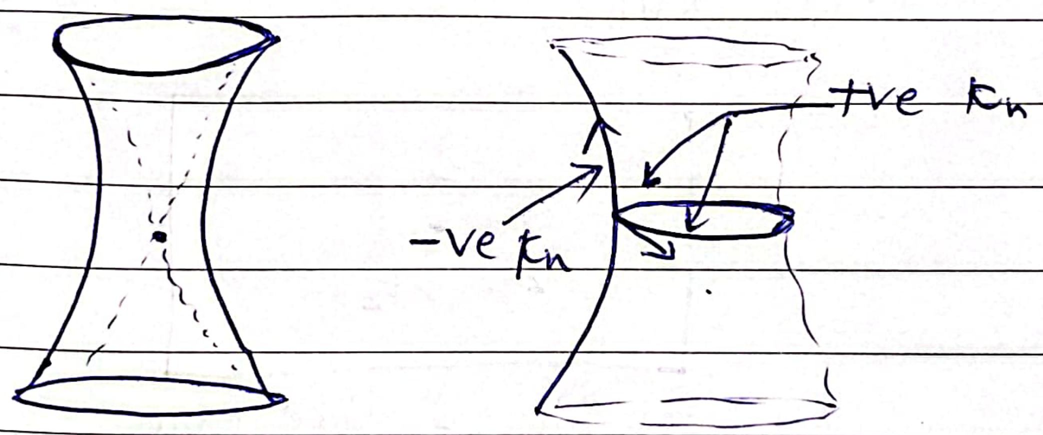
$= k_1 \cos^2\theta + k_2 (1-\cos^2\theta)$

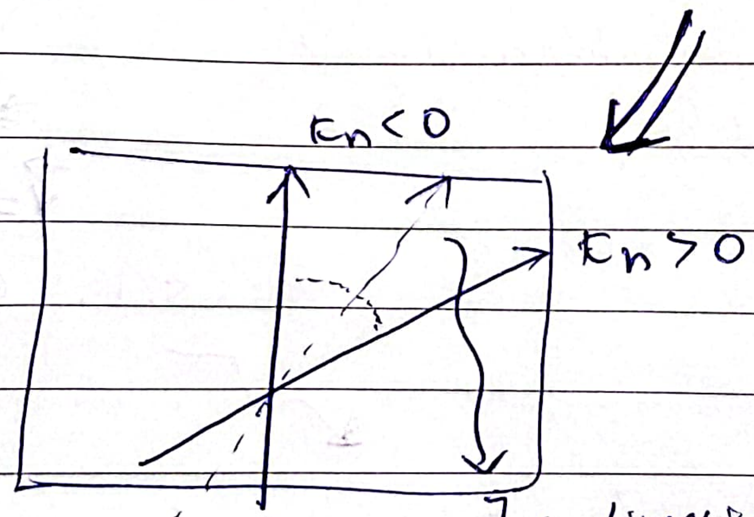
$= (k_1 - k_2)\cos^2\theta + k_2$

If $k_2 > k_1$, $k_n - k_2 \leq 0$

$\Rightarrow \boxed{k_n \leq k_2}$

Thus k_1 is the smallest possible k_n (at $\theta = 0$) and k_2 is the largest ($\theta = \pi/2$).





$\exists$ a direction where $\underbrace{k_n = 0}$

the direction where the lines exist.

for curves

$$\underbrace{\hat{T}(t)}_{\text{tangent}}$$

curvature $\kappa(t) := \|\hat{T}'(t)\|$

for surfaces

$$\tilde{n}(x, y)$$

gives the tangent plane

curvature in $\vec{v}$ direction

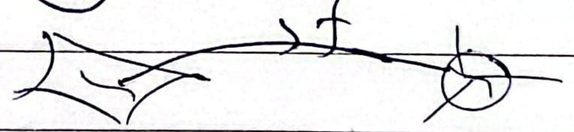
$$\underbrace{[D_p \tilde{n}]\vec{v}}_{\text{change of } \tilde{n} \text{ in } \vec{v} \text{ direction.}}$$

OR

$$= \vec{v} \cdot W_p(\vec{v})$$

$\boxed{\text{NEXT}}$ ~~Thm~~

Something called isometries between surfaces,



$$f : S_1 \longrightarrow S_2$$

is called distance-preserving if for all curves on S_1

$$\gamma_1 : (a, b) \longrightarrow S_1$$

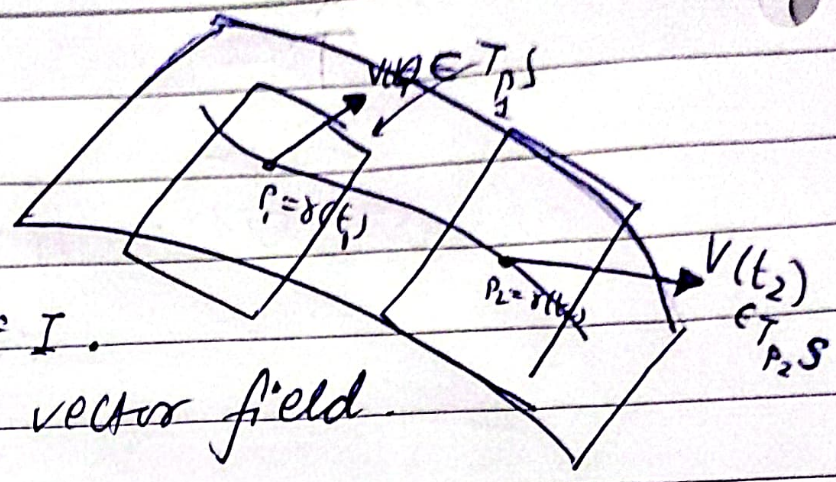
$$\int_{[t_1, t_2]} \|\dot{\gamma}_1\| = \int_{[t_1, t_2]} \|(f \circ \gamma_1)'\|$$

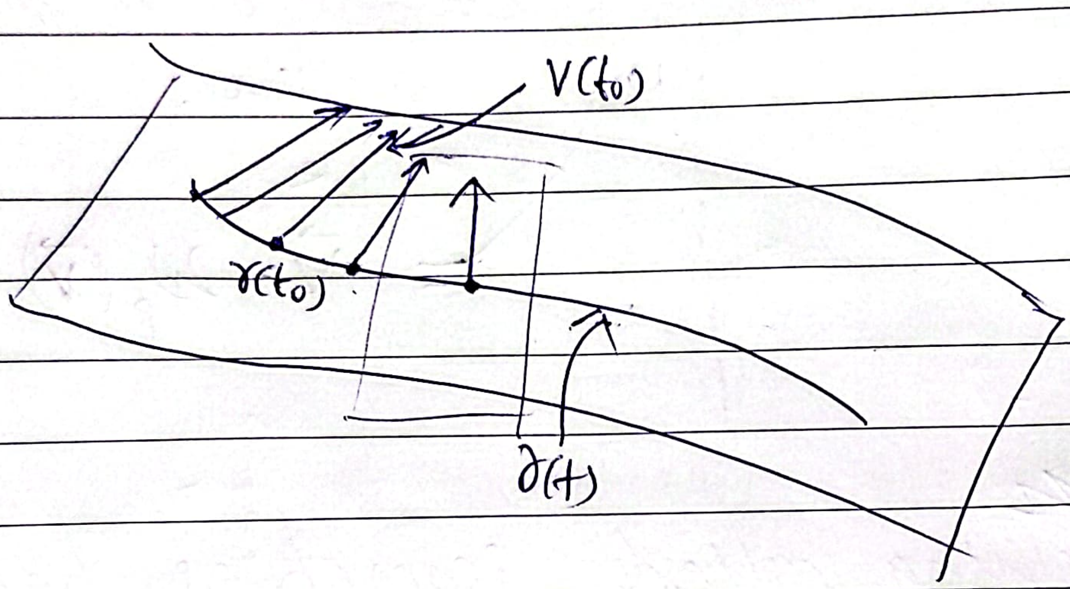
AKA local Isometry.

"Parallel vectors" on surfaces

21 NOV 22

(Defn) Given a surface S and a curve $\gamma : I \to S$
we have a function $V : I \to \{ ? \}$
s.t. $V(t) \in T_{\gamma(t)} S$ for all $t \in I$.
Then V is called a tangent vector field.





(Example) Given $\gamma(t) : I \to S$
$\dot{\gamma}(t)$ is a tangent vector field.

$$V(t) := \dot{\gamma}(t)$$

then $\dot{V}(t) = \ddot{\gamma}(t)$ ← need not be tangent.
and no notion of "pallel ⇒ zero derivative"

So, the "new notion":

$$\underbrace{\nabla_\gamma (V)}_{\text{covariant derivative (along curve } \gamma \text{ of vector field } V)} = \underbrace{\dot{V}(t) - \dot{V} \cdot \hat{n}(\gamma(t))\, \hat{n}(\gamma(t))}_{\text{component of } \dot{V}(t) \text{ along } T_{\gamma(t)} S}$$

$v(t) := \dot\gamma(t) \longleftarrow$ unit speed param. curve γ Δ

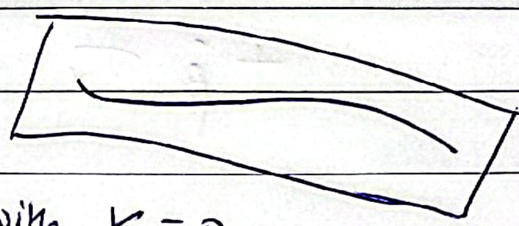
* Notice:

$$\|\nabla_\gamma(\dot\gamma)\| \longleftarrow \kappa_g(t)$$

(Defn)

* When $\nabla_\gamma(v(t)) \equiv 0$ for all t, then $v(t)$ is said to be <u>parallel along $\gamma(t)$</u>.

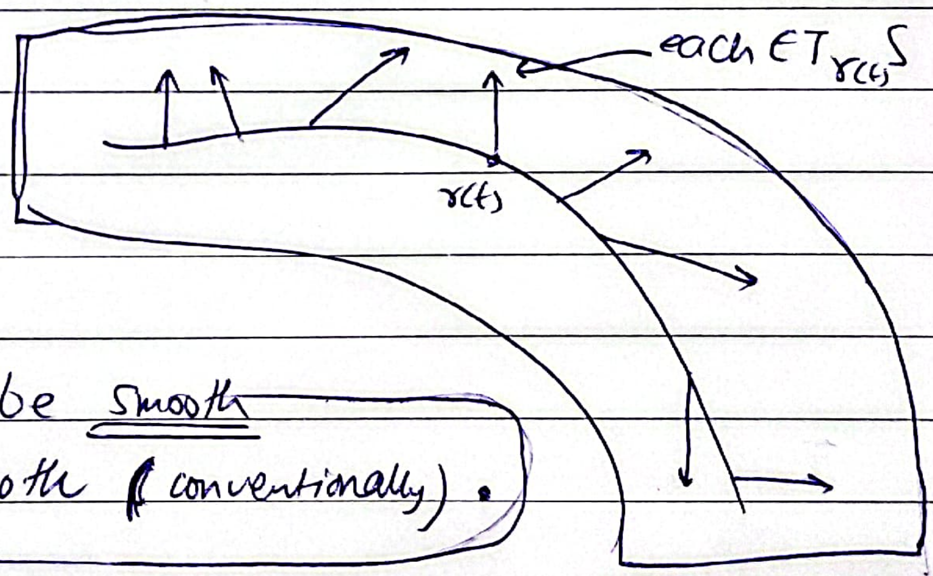
{ Paths on surfaces with the shortest length locally → also will produce curves with $\kappa_g \equiv 0$. } → geodesics



Say $\gamma(t) := \sigma(\delta(t))$,

then a general tangent vector field will look like:

$$V(t) = \alpha(t)\,\sigma_x(\delta(t)) + \beta(t)\,\sigma_y(\delta(t))$$



(Defn) $V(t)$ is said to be <u>smooth</u> if $\alpha(t)$, $\beta(t)$ are smooth (conventionally).

(Claim) The definition remains well-defined if we change σ

→ Use Inverse function theorem!

NOW,

$$\nabla_\gamma(V(t)) = \underbrace{\dot{V} - (\dot{V}\cdot\hat{n})\,\hat{n}}$$

$$V(t) = \alpha(t)\,\sigma_x\big|_{\delta(t)} + \beta(t)\,\sigma_y\big|_{\delta(t)}$$

$$\dot{V} = \dot{\alpha}\,\sigma_x + \alpha\left(\delta_x'\,\sigma_{xx} + \delta_y'\,\sigma_{xy}\right)$$
$$+\,\dot{\beta}\,\sigma_x + \beta\left(\delta_x'\,\sigma_{yx} + \delta_y'\,\sigma_{yy}\right)$$

$$\begin{cases} \sigma_{xx} = \begin{bmatrix} \Gamma_{11}^1 \\ \\ \\ \end{bmatrix}\sigma_x + \begin{bmatrix} \Gamma_{11}^2 \\ \\ \\ \end{bmatrix}\sigma_y + \begin{bmatrix} \\ \\ \\ \end{bmatrix}\hat{n} \\ \sigma_{xy} = \\ \sigma_{yy} = \end{cases}$$

$\nabla_\gamma(V)$ = ignore $\hat{n}$ part from $\dot{V}(t)$. ✓

26OCT22

Quiz 3

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5/8

Page 1/2

Consider a sphere of radius r,

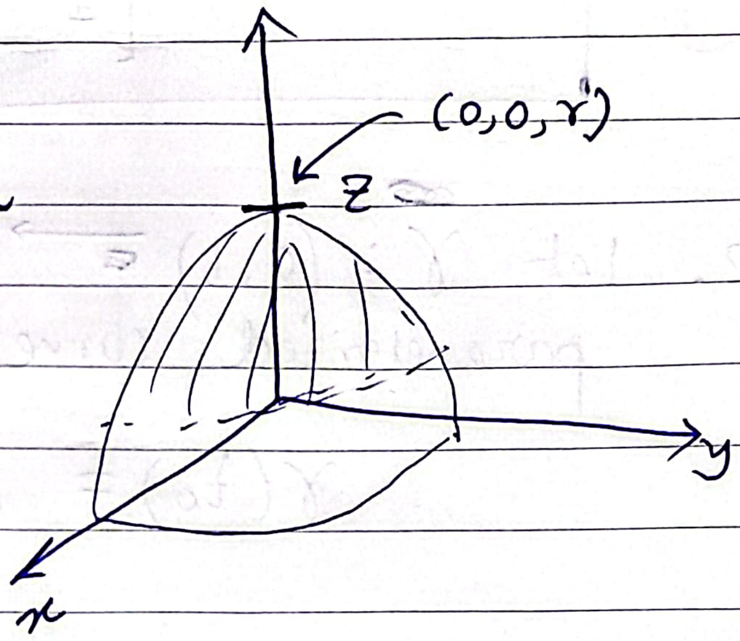
$$S := \{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = r^2 \}$$

1. Compute unit normal to sphere at $(0, 0, r)$. (2)
2. Compute the normal curvature of any unit speed parameterization passing through the point $(0,0,r)$ (3)

ANSWERS

We take a surface patch (U, σ) such that

$$(0,0,r) \in \sigma(U).$$



We take the σ to be

$$\sigma : (0,r)\times(0,r)\cancel{\times(0,r)} \to \mathbb{R}^3$$

$$\sigma(x, y) = (x, y, \sqrt{1 - x^2 - y^2})$$

Which is a smooth, regular surface patch.

~~1.~~ $\cancel{\sigma_x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}$

1. $\sigma_x = \begin{bmatrix} 1 \\ 0 \\ \dfrac{1}{2\sqrt{1-x^2-y^2}} \cdot (-2x) \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 \\ 1 \\ \dfrac{1}{2\sqrt{1-x^2-y^2}} (-2y) \end{bmatrix}$

at $(0,0,r)$, $\sigma_x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\sigma_y = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

normal at $(0,0,r)$

$$\hat{n}\Big|_{(0,0,r)} := \frac{\sigma_x \times \sigma_y}{\|\sigma_x \times \sigma_y\|} = \frac{\begin{bmatrix}1\\0\\0\end{bmatrix} \times \begin{bmatrix}0\\1\\0\end{bmatrix}}{\left\|\begin{bmatrix}1\\0\\0\end{bmatrix} \times \begin{bmatrix}0\\1\\0\end{bmatrix}\right\|}$$

$$\boxed{\hat{n}\Big|_{(0,0,r)} = \begin{bmatrix}0\\0\\1\end{bmatrix}}$$

(it could be $\pm\begin{bmatrix}0\\0\\1\end{bmatrix}$ by some different σ).

2. Let $\gamma : (a,b) \longrightarrow S$ to be a unit speed parameterized curve such that

$$\gamma(t_0) = (0,0,r)$$

where $t_0 \in (a,b)$.

Normal curvature, $k_n(t) := \gamma''(t)\cdot\hat{n}(\gamma(t))$

Here, at $(0,0,r)$ the normal curvature is

$$k_n(t_0) = \gamma''(t_0)\cdot\hat{n}(\gamma(t_0))$$

Now, as γ lies on a sphere

$\boxed{\gamma\cdot\gamma = r^2}$, and $\gamma(t_0) = r\begin{bmatrix}0\\0\\1\end{bmatrix} = r\,\hat{n}\Big|_{(0,0,r)}$.

$\Rightarrow \gamma\cdot\gamma' = 0$

$\Rightarrow \gamma\cdot\gamma'' + \gamma'\cdot\gamma' = 0$

$\Rightarrow \gamma''\cdot\gamma = -\|\gamma'\|^2 = -1 \quad [\|\gamma'\| = 1]$.

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Hence,
$$\left.(\gamma'' \cdot \gamma)\right|_{t_0} = -1$$

and
$$\gamma(t_0) = r\hat{n}(0,0,r)$$

$$\Rightarrow \quad \gamma'' \cdot (r\hat{n})_{(0,0,r)} = -1$$

As
$$k_n(t_0) = \gamma''(t_0) \cdot \left.\hat{n}\right|_{(0,0,r)}.$$

$$\Rightarrow \quad \boxed{k_n(t_0) = \frac{-1}{r}}$$

Hence, normal curvature thus computed is $-\frac{1}{r}$, with $\hat{n} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

We could have also chosen $-\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ as normal which would give $+\frac{1}{r}$ as the curvature.

$$\boxed{\text{Hence, the normal curvature at } (0,0,r) \text{ of any curve is } \pm\frac{1}{r}.}$$

3

Quiz 4
MTH204

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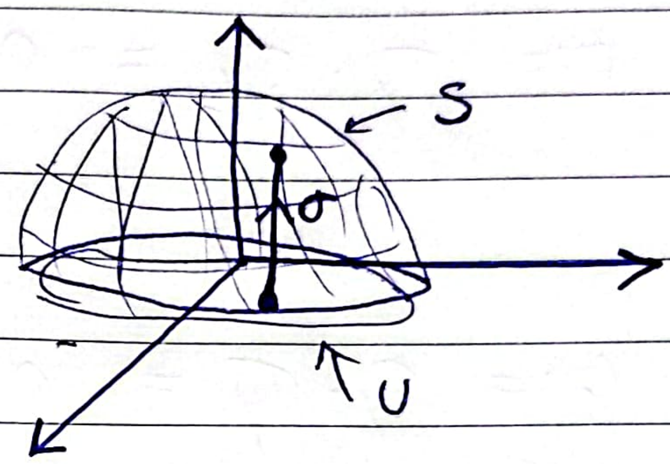
Consider $S := \{ (x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1 \}$

(1) Give it a surface patch
(2) Use it to compute first fundamental form: E, F, G

——— × ———

(1) We take the surface patch to be



$U := \{ (x,y) \in \mathbb{R}^2 \mid x^2+y^2 < 1 \}$

$\sigma : U \longrightarrow S$

$\sigma(x,y) = (x, y, \sqrt{1-x^2-y^2})$

this covers the upper hemisphere (without equator) and we can take multiple such σ to cover the whole sphere, which will all be similar projections.

We know it is a correct surface patch because,

$$\sigma(x,y) = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ \sqrt{1-x^2-y^2} \end{bmatrix}$$

Here $\sigma_1^2 + \sigma_2^2 + \sigma_3^2 = 1$, hence, $\sigma(x,y) \in S$, and it is a homeomorphism.
And we also know σ is smooth and regular here.

(2) To compute the first fundamental form we compute:

$$\sigma_x = \begin{bmatrix} 1 \\ 0 \\ \frac{-x}{\sqrt{1-x^2-y^2}} \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 \\ 1 \\ \frac{-y}{\sqrt{1-x^2-y^2}} \end{bmatrix}$$

Now, $E := \sigma_x \cdot \sigma_x = 1 + 0 + \dfrac{x^2}{1-x^2-y^2}$

$$= \frac{1-y^2}{1-x^2-y^2}$$

$$F := \sigma_x \cdot \sigma_y = 0 + 0 + \frac{xy}{1-x^2-y^2}$$

$$G := \sigma_y \cdot \sigma_y = 0 + 1 + \frac{y^2}{1-x^2-y^2}$$

$$= \frac{1-x^2}{1-x^2-y^2}$$

Hence, the matrix

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

$$= \frac{1}{1-x^2-y^2}\begin{pmatrix} 1-y^2 & xy \\ xy & 1-x^2 \end{pmatrix}$$

2 3

And similarly, we use other surface patches to cover the whole sphere and compute the form, which ~~could~~ can be similarly done.

Terminology	Notation	Definition
First fundamental form	E	$E = \frac{\partial f}{\partial u} \cdot \frac{\partial f}{\partial u}$
	F	$F = \frac{\partial f}{\partial u} \cdot \frac{\partial f}{\partial v}$
	G	$G = \frac{\partial f}{\partial v} \cdot \frac{\partial f}{\partial v}$
Second fundamental form	L	$L = \frac{\partial^2 f}{\partial u^2} \cdot n$
	M	$M = \frac{\partial^2 f}{\partial u \partial v} \cdot n$
	N	$N = \frac{\partial^2 f}{\partial v^2} \cdot n$
[redacted]		
Gaussian curvature	K	$K = \frac{LN - M^2}{EG - F^2}$
Mean curvature	H	$H = \frac{GL - 2FM + EN}{2(EG - F^2)}$
Principal curvatures	$\kappa_{\pm}$	$H \pm \sqrt{H^2 - K}$

Geodesics

γ be a curve on S. $(\gamma : I \longrightarrow S)$

$\nabla_\gamma(\dot{\gamma}) \equiv 0 \quad \forall t.$

Alternatively,

$$\gamma : I \longrightarrow S \text{ (surface)}$$

is a geodesic iff its acceleration is either $\vec{0}$ or perpendicular to ~~the~~ tangent space.

$\boxed{\text{Example}}$ (Exercise)

(1) If $\ddot{\gamma} \equiv 0$, (st. line). then it is a geodesic.

(2) On a plane, any geodesic is a st. line.

(3) geodesic $\iff$ $\kappa_g \equiv 0$.

$\gamma : I \longrightarrow S$ (surface) is a geodesic.

Given, $\sigma : U \longrightarrow S$ (smooth, regular)

$$\gamma(t) = \sigma(x(t), y(t))$$

$$\dot{\gamma} = \dot{x}\sigma_x + \dot{y}\sigma_y$$

$$\ddot{\gamma} = \ddot{x}\sigma_x + \dot{x}\,\underbrace{\cancel{\sigma_{xx}}}_{(\dot{x}\sigma_{xx} + \dot{y}\sigma_{xy})} + \ddot{y}\sigma_y + \dot{y}(\dot{x}\sigma_{yx} + \dot{y}\sigma_{yy})$$

So, $\nabla_\gamma(\dot{\gamma}) \equiv 0 \iff \ddot{\gamma}\cdot\sigma_x \equiv 0$ and $\ddot{\gamma}\cdot\sigma_y \equiv 0$

$$(\dot{x}\sigma_x + \dot{y}\sigma_y)' \cdot \sigma_x = 0$$

and $\Rightarrow$ $$(\dot{x}\sigma_x + \dot{y}\sigma_y)' \cdot \sigma_x = 0.$$

$$\Rightarrow \left((\dot{x}\sigma_x + \dot{y}\sigma_y) \cdot \sigma_x\right)'$$

$$= \cancel{(\dot{x}\sigma_x + \dot{y}\sigma_y)'}^{\,0} \cdot \sigma_x$$

$$+ (\dot{x}\sigma_x + \dot{y}\sigma_y) \cdot \underbrace{(\sigma_x(x(t), y(t)))'}$$

$$\Rightarrow (\dot{x}E + \dot{y}F)' = (\dot{x}\sigma_x + \dot{y}\sigma_y) \cdot (\dot{x}\sigma_{xx} + \dot{y}\sigma_{xy})$$

$$= \dot{x}^2 \underbrace{\sigma_x \cdot \sigma_{xx}}_{\frac{E_x}{2}} + \dot{x}\dot{y}(\sigma_x \cdot \sigma_{xy} + \sigma_y \cdot \sigma_{xx})$$

$$+ \dot{y}^2 \sigma_y \cdot \sigma_{xy}$$

$\vdots$

a ~~first~~ second order ODE

for $x(t)$, $y(t)$.

28NOV22

⋮

$$(E\dot{x} + F\dot{y})' = \frac{1}{2}\left(E_x \dot{x}^2 + 2F_x \dot{x}\dot{y} + G_x \dot{y}^2\right)$$

$$(F\dot{x} + G\dot{y})' = \frac{1}{2}\left(E_y \dot{y} + 2F_y \dot{y}\dot{x} + G_y \dot{y}^2\right)$$

given $\dot{x}(0)$, $x(0)$ $\longrightarrow$ $\exists!\ \dot{x}(t)$ ~~for~~ as solution of this equation.

$\dot{y}(0)$, $y(0)$ $\qquad \dot{y}(t)$

$p = \sigma(x(0), y(0))$ there exist unique

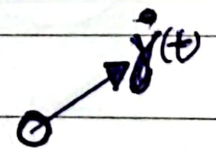
$\vec{v} := \dot{x}(0)\sigma_x(p) + \dot{y}(0)\sigma_y(p) \in T_pS$

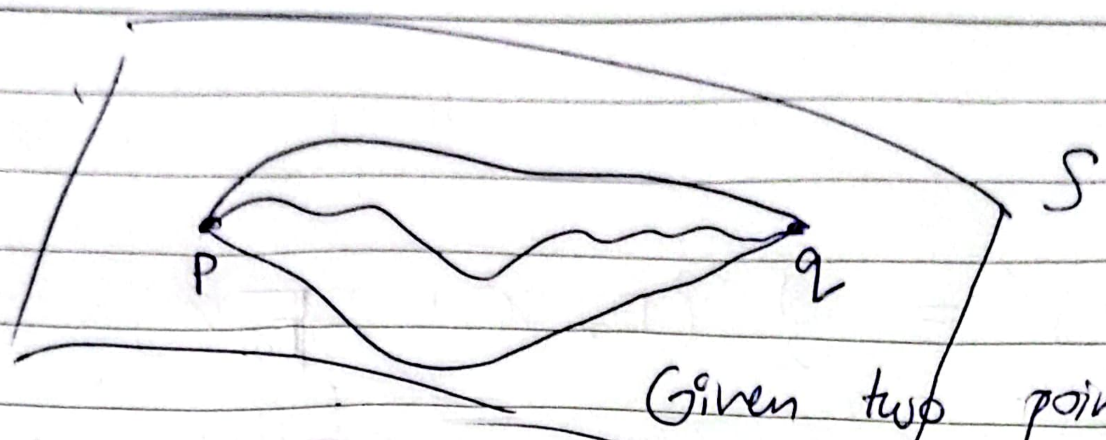
$\Longrightarrow$ exists a $\boxed{\text{unique}}$ geodesic γ passing through p ($\gamma(0) = p$) and $\dot{\gamma}(0) = \vec{v}$

"local"

~~[illegible]~~

while





Given two points $q, p \in S$, what is the curve $\gamma : \boxed{[0,1]} \to S$ (Unit speed) s.t. $\gamma(0) = p$, $\gamma(1) = q$, has "the minimum length property"

$$\int_{t=0}^{t=1} \|\dot{\gamma}\|$$

Say $F : (-\epsilon, \epsilon) \times [0,1] \longrightarrow S$

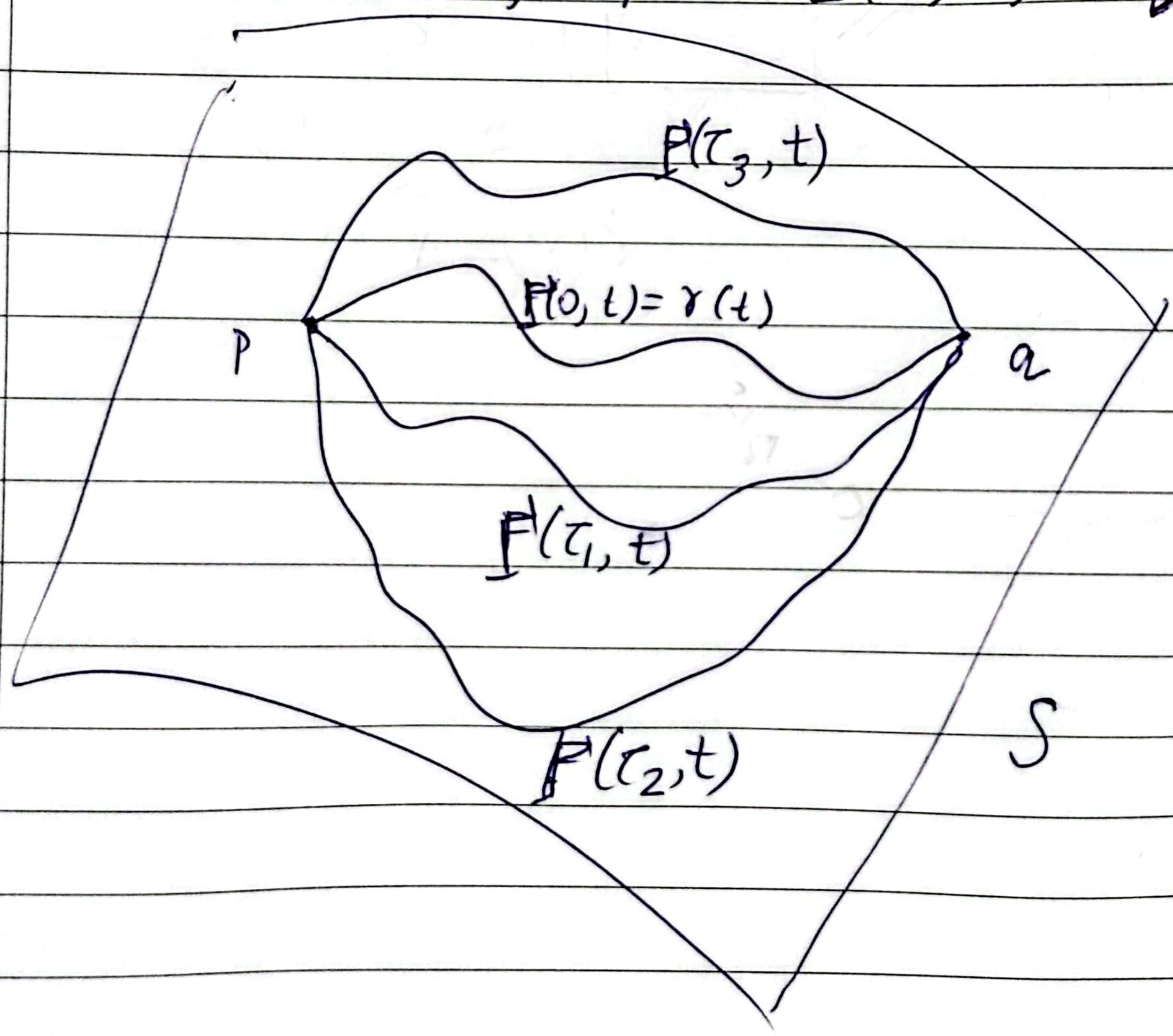
is smooth ~~the way we define it~~ :

and :

$$F(0,t) := \gamma(t)$$

and for any t

$$F(\tau, 0) = p \,, \quad F(\tau, 1) = q$$



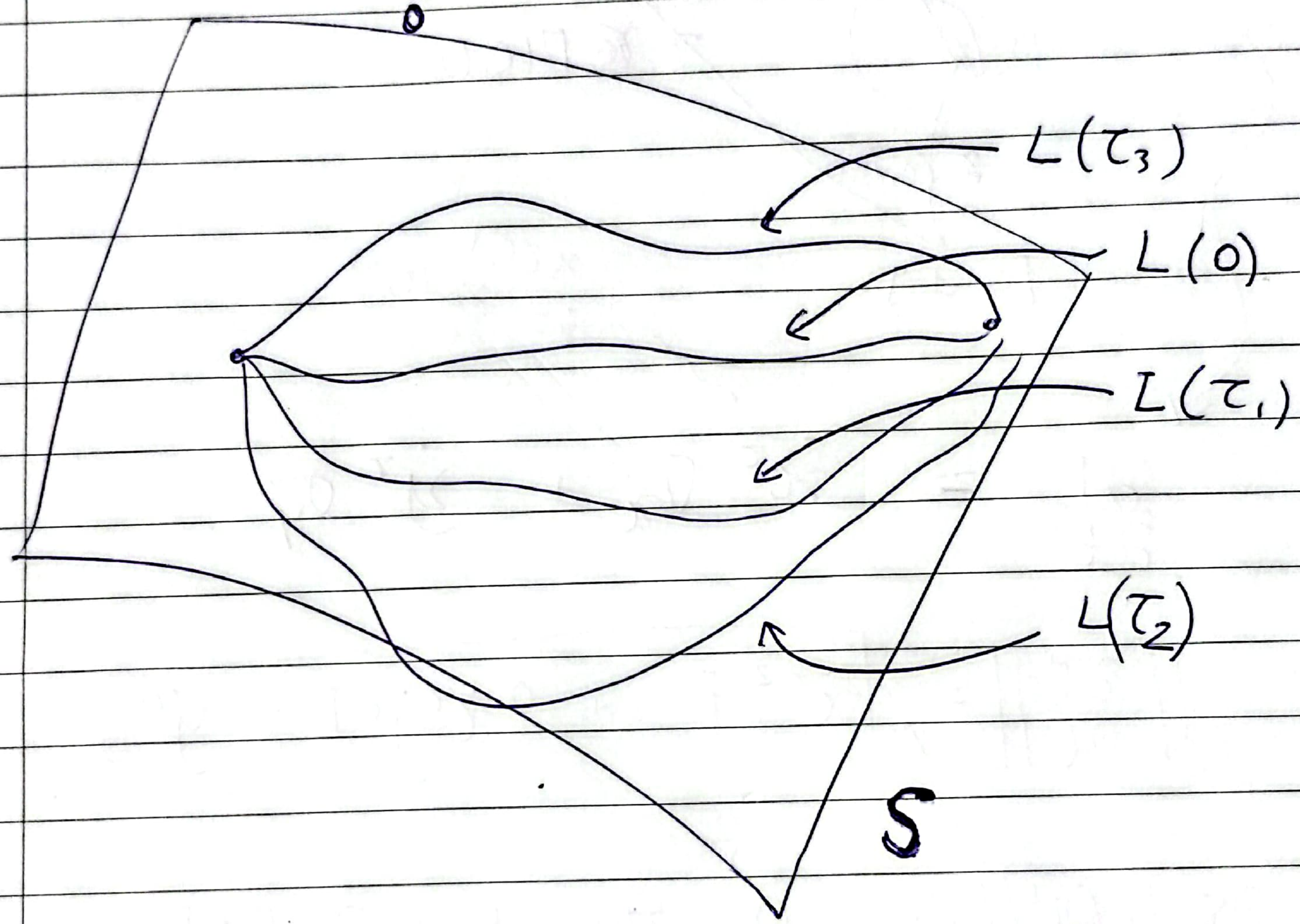
But

We do not choose any explicit Γ! No matter which Γ we choose the results will remain the same.

(Book Notation : $\Gamma(\tau, t) = \underbrace{\gamma^{\tau}(t)}_{\text{book}}$)

Now,

$$L(\tau) = \int_0^1 \left\|\frac{\partial \Gamma}{\partial t}(\tau, t)\right\| dt$$



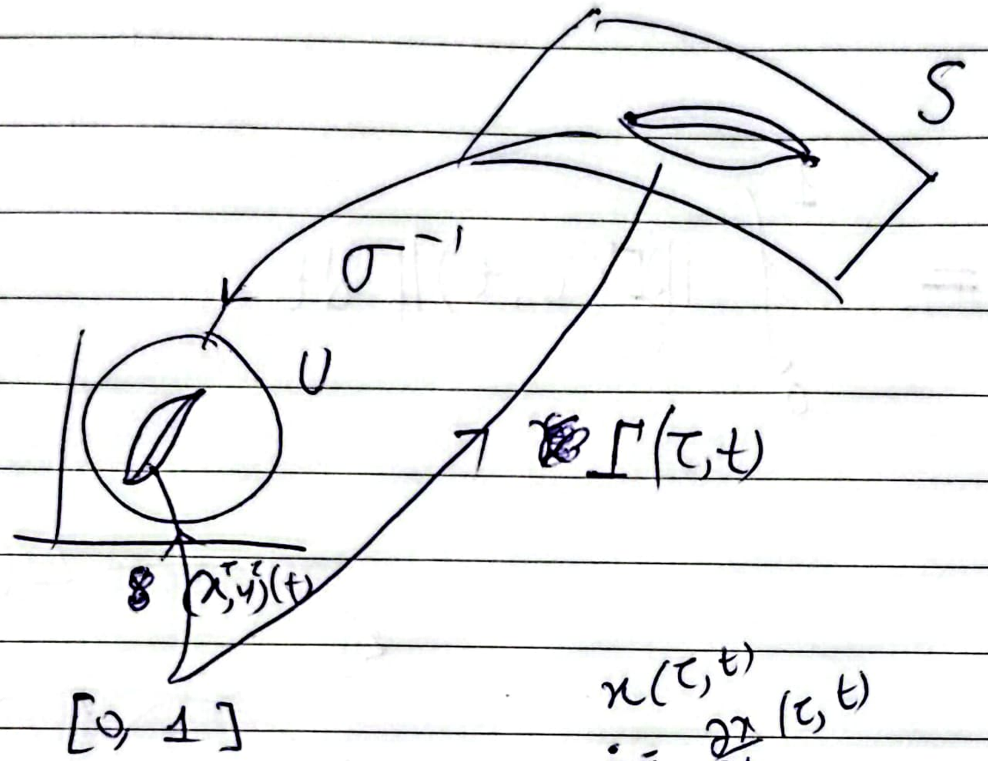
So, the extremum is at $\dfrac{dL}{d\tau} = 0$

$$\Rightarrow \lim_{h\to 0} \frac{\frac{\partial L}{\partial t}(\tau+h) - \frac{\partial L}{\partial t}(\tau)}{h} = \lim_{h\to 0} \frac{\int \frac{\partial \Gamma}{\partial t}(\tau+h)\,dt - \int \frac{\partial \Gamma}{\partial t}(\tau)\,dt}{h} = 0$$

$$\Rightarrow \frac{dL_0}{d\tau} = \int_0^1 \frac{\partial \left\|\frac{\partial \Gamma}{\partial t}(\tau, t)\right\|}{\partial \tau}\,dt = 0$$

$$\Rightarrow \int_0^1 \frac{\partial}{\partial \tau} \left\| \frac{\partial \Gamma}{\partial t}(\tau, t) \right\| dt = 0$$

$$\Gamma(\tau, t) = \sigma(x^\tau(t), y^\tau(t))$$



$x(\tau, t)$

$\dot{x} \equiv \frac{\partial x}{\partial t}(\tau, t)$

$$\frac{\partial \Gamma}{\partial t} = \dot{x}^\tau \sigma_x + \dot{y}^\tau \sigma_y$$

$$\left\| \frac{\partial \Gamma}{\partial t} \right\| = \sqrt{\dot{x}^{\tau 2} E + 2 \dot{x}^\tau \dot{y}^\tau F + \dot{y}^{\tau 2} G}$$

$$\int_0^1 \left(\frac{\partial}{\partial \tau} \left\| \frac{\partial \Gamma}{\partial t} \right\| \right) dt = 0$$

Then the $\frac{\partial}{\partial \tau} \left\| \frac{\partial \Gamma}{\partial t} \right\|$ has to be 0 if we are ~~not choosi~~ saying no matter what Γ is, this equation should.

$\vdots$

essentialy we will get same equations as the previous method!

Note:
$\exists\, \Gamma(\tau,0)=p,\ \Gamma(\tau,1)=q$

$\frac{\partial\Gamma}{\partial\tau}\left[\begin{smallmatrix}\tau\\0\end{smallmatrix}\right] = 0 = \frac{\partial\Gamma}{\partial\tau}\left[\begin{smallmatrix}\tau\\1\end{smallmatrix}\right]$

(Defn) Let $f : S_1 \longrightarrow S_2$ be smooth.
s.t. "f preserves arc-length"

Given a $\gamma : (a,b) \longrightarrow S_1$

$$\int_{[a,b]} \|(f\circ\gamma)'\| = \int_{[a,b]} \|\dot{\gamma}\|$$

for all γ.

then f is called an isometry

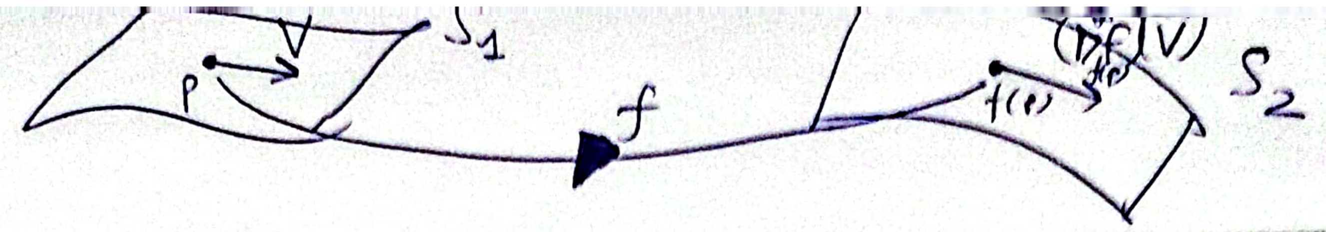
"length preserving fn b/w surfaces"

$$(f\circ\gamma)' = \underbrace{(D^*_{\gamma(t)} f)}(\dot{\gamma}(t))$$

$$D^*_p f : T_p S_1 \xrightarrow{\sim} T_{f(p)} S_2$$

linear

" s.t. it converts velocities "



$$\Leftrightarrow \int_{[a,b]} \|(\underbrace{D^*_{\gamma(t)} f}) (\dot{\gamma})\| = \int_{[a,b]} \|\dot{\gamma}\| \quad \text{(length preserving)}$$

$$\Leftrightarrow \text{for any } p \in S, \ v \in T_p S_1$$

$$\|(D^*_p f)(\vec{v})\| = \|\dot{\gamma}\|$$

(norm of tangent vector preserving)

because

$(\|D^* f(\dot{\gamma})\| - \|\dot{\gamma}\|)$ is continious

take ~~choose ε~~ $\displaystyle\int_{[t_0+\epsilon,\, t_0+\epsilon]} (\underbrace{\|D^* f(\dot{\gamma})\| - \|\dot{\gamma}\|}) = 0$

choose ϵ such that this is only +ve or −ve.

$$\Rightarrow (\|(D^* f)(\dot{\gamma})\| - \|\dot{\gamma}\|) \equiv 0 \ \cancel{\text{for all}}$$

at t_0

for any $\dot{\gamma}$

Hence, because the reverse implication is also true, it's iff.

Isometry

Previously,

If

$f: S_1 \to S_2$

smooth, bijective

s.t. for any $\gamma: I \to S_1$

$\int \|\dot{\gamma}\| = \int \|(f\circ\gamma)'\|$ $\Leftrightarrow$ $\|(D_p^* f)(v)\| = \|\vec{v}\|$ $\forall p \in S,\ v \in T_p S_1$

$(D_p^* f)$ ← "preserving tangent vector norms."

" preserving arc-lengths "

Polar Identity:

$$\|\vec{x}+\vec{y}\|^2 = \|x\|^2 + \|y\|^2 + 2\vec{x}\cdot\vec{y}$$

$$\Rightarrow \vec{x}\cdot\vec{y} = \frac{\|\vec{x}+\vec{y}\|^2 - \|x\|^2 - \|y\|^2}{2}$$

If $\cancel{D_p \forall}$ $\|D_p f(\vec{v})\| = \|\vec{v}\|$ $\forall p \in S,\ v \in T_p S$

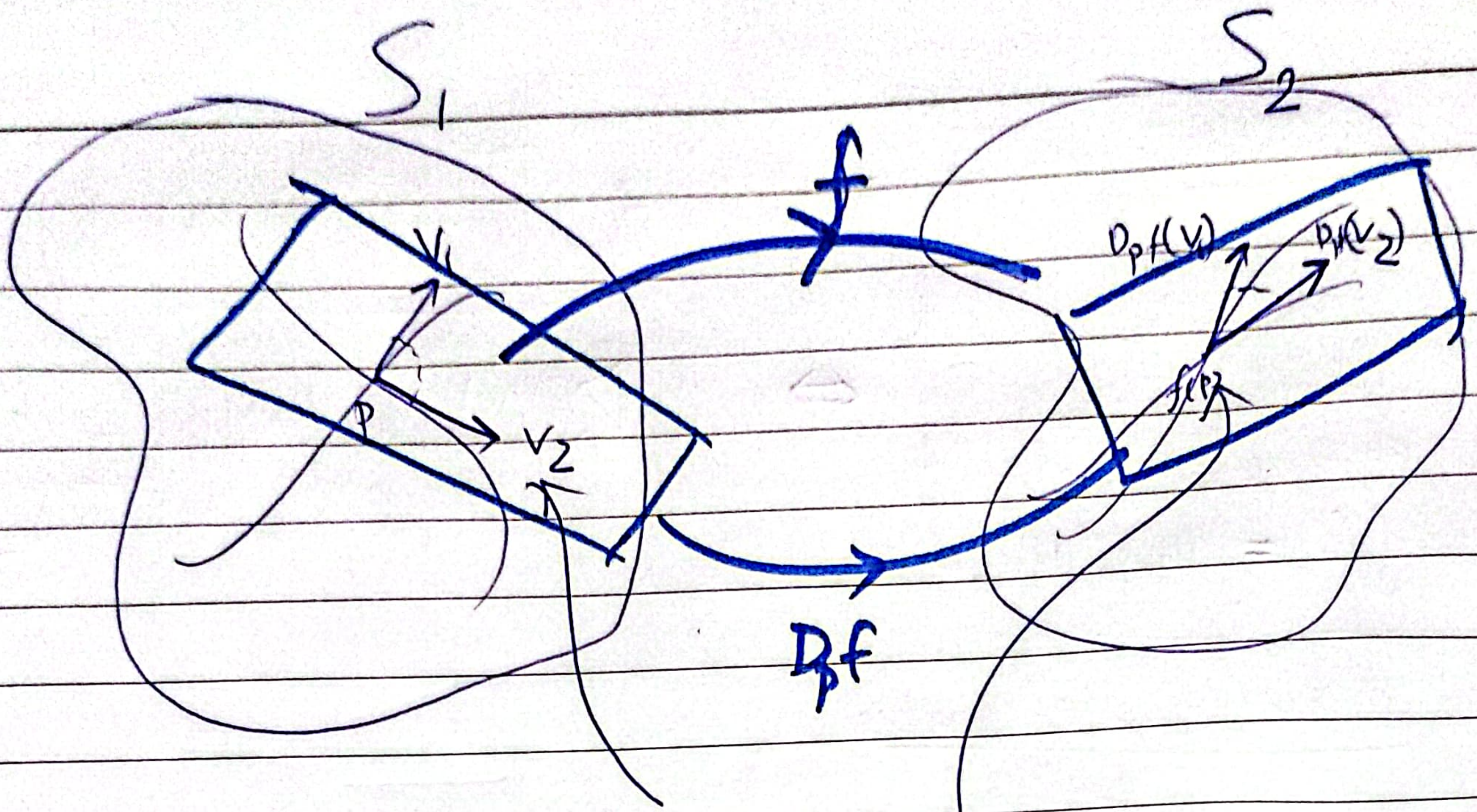
$\Rightarrow$ $(D_p f)(v_1)\cdot(D_p f)(v_2)$ $\forall v_1, v_2 \in T_p S$

$$= \frac{\|D_p f(v_1+v_2)\|^2 - \|D_p f(v_1)\|^2 - \|D_p f(v_2)\|^2}{2}$$

$$= \frac{\|v_1+v_2\|^2 - \|v_1\|^2 - \|v_2\|^2}{2}$$

$$= v_1 \cdot v_2$$

$D_p f(v)$ also preserves the dot-products.



$$\underbrace{\frac{v_1 \cdot v_2}{\|v_1\|\|v_2\|} = \frac{(D_pf)(v_1) \cdot (D_pf)(v_2)}{\|D_pf\, v_1\| \, \|D_pf(v_2)\|}}$$

(D_pf) preserve angles too!

$v, w \in T_pS$

$$\langle \vec{v}, \vec{w} \rangle^{FIRST}_{FFS} = \vec{v} \cdot \vec{w}$$

(first f.f. is just dot product!)

only constrainted to T_pS

If we have a $f : S_1 \longrightarrow S_2$, a diffeomorphism, (f bijective, smooth f^{-1} also smooth) ,

we have ~~[illegible]~~ $(D_pf) : T_pS_1 \longrightarrow T_{f(p)}S_2$

$v, w \in$

s.t.

$$v \cdot w = (D_pf)(\vec{v}) \cdot (D_pf)(\vec{w})$$

if f is an isometry

first f.f. of S_2

but in other words:

define "pullback by f" as

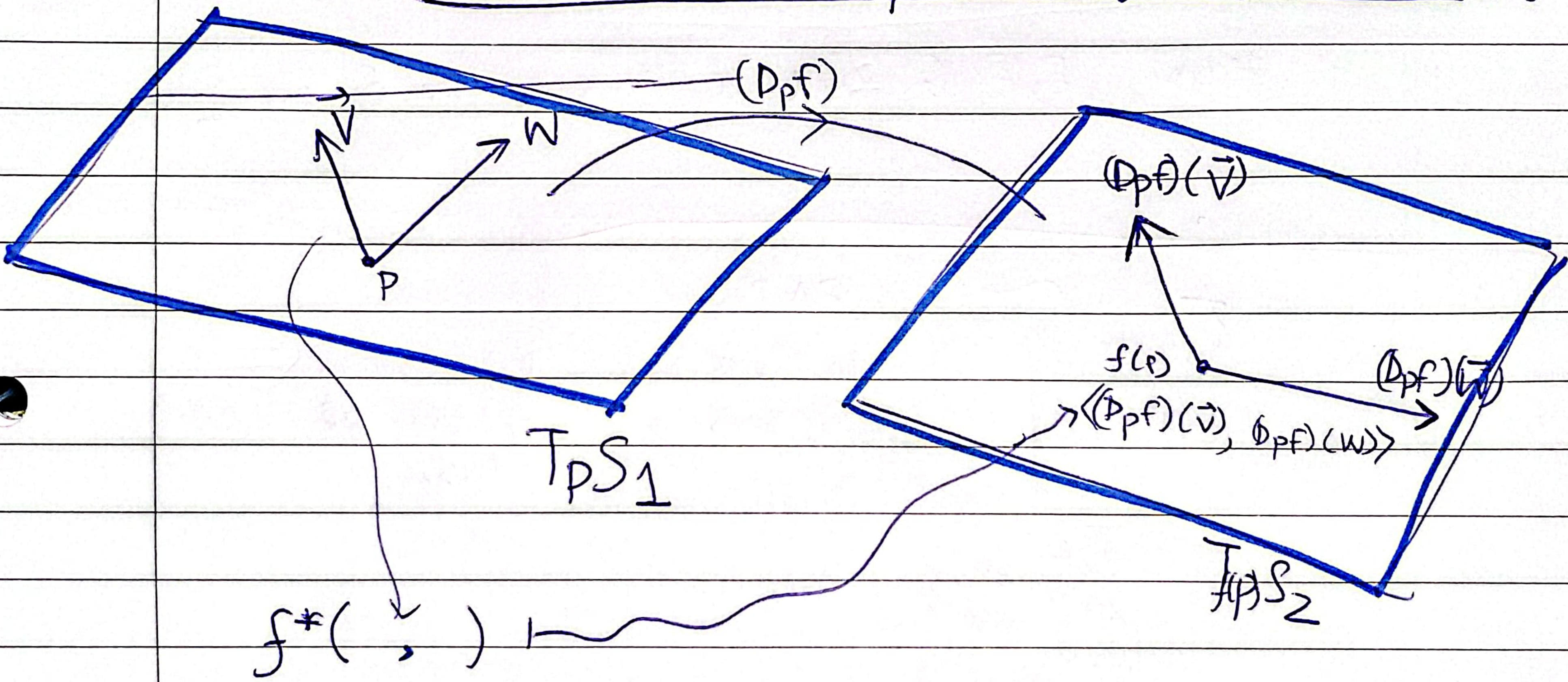
$$f^*(\vec{v},\vec{w}) := \langle D_pf(v), D_pf(w)\rangle^{\text{FIRST}}_{f(p)\in S_2}$$

for each $v, w \in T_pS_1$
$\forall p \in S_1$

$$f^* : T_pS_1 \times T_pS_1 \longrightarrow \mathbb{R}$$

f^* is a "form" on T_pS_1 !

but it uses D_pf and first f.f. of S_2!



Now, if f is an isometry, then

$$f^*(\vec{v},\vec{w}) = \langle v, w\rangle^{\text{FIRST}}_{p\in S_1}$$

$$\Big\{ \text{OR just simply } \langle D_pf(v), D_pf(w)\rangle^{\text{FIRST}}_{f(p)\in S_2} = \langle v, w\rangle^{\text{FIRST}}_{p\in S_1} \Big\}$$

$$\langle D_p f(c_1\sigma_x + c_2\sigma_y) ; D_p f(\lambda_1\sigma_x + \lambda_2\sigma_y)\rangle$$
$$= \lambda_1 c_1 \, [D_p f(\sigma_x)]^2 + (\lambda_1 c_2 + \lambda_2 c_1)\, D_p f\,\sigma_x \cdot D_p f\,\sigma_y$$
$$+ \; c_2\lambda_2((D_p f)\sigma_y)^2$$

$$\begin{tikzcd} S_1 \arrow[r, "f"] & S_2 \\ U_1 \arrow[u, "\sigma_1"] \arrow[ur, dashed, "\sigma_2 := (f\circ\sigma_1)"'] & \end{tikzcd}$$

$S_1 \xrightarrow{f} S_2$ be a isometry (smooth, bij)

then if $\sigma_2 := (f\circ\sigma_1) : U_1 \longrightarrow S_2$

then E, F, G will be the same for S_1 and S_2.

$A \sim PAP^{-1}$

A in some basis

PAP^{-1} in some other basis $\longleftarrow$ P is change of basis matrix

the matrices are called "similar"