Info

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$\mathbb{C}$-measure

Definition. $\mathbb{C}$-measures on a measure space

Let $\mathfrak{M}$ be a σ-algebra on a set X. A $\mathbb{C}$**-measure** on $\mathfrak{M}$ is a function

$$\mu : \mathfrak{M} \to \mathbb{C}$$

such that

$$\{E_i\} \subseteq \mathfrak{M} \text{ partitions } E \in \mathfrak{M} \implies \mu(E) = \sum_{i\geq 1} \mu(E_i)$$

- The **convergence of** $\sum_{i\geq 1} \mu(E_i)$ is now a requirement for every partition of E, unlike for positive measures where the series could converge or diverge to ∞.
- Because $E = \bigcup_{i\geq 1} E_i$ does not change under a rearrangement of (E_i) the series $\sum_{i\geq 1} \mu(E_i)$ must actually **converge absolutely**.

dominating a complex measure with positive measures

Any positive measure $(X, \mathfrak{M}, \lambda)$ that dominates a complex measure $(X, \mathfrak{M}, \mu)$, meaning

$$E \in \mathfrak{M} \implies |\mu(E)| \leq \lambda(E)$$

must have

$$\{E_i\} \text{ partitions } E \in \mathfrak{M} \implies \lambda(E) = \sum_{i\geq 1} \lambda(E_i) \geq \sum_{i\geq 1} |\mu(E_i)|$$

So $\lambda(E)$ is at least equal to the supremum of all possible sums on the right that is

Definition. The **total variation** of a complex measure $(X, \mathfrak{M}, \mu)$ is

$$\begin{aligned} |\mu| : \mathfrak{M} &\to [0, \infty) \\ E \mapsto |\mu|(E) &:= \sup_{\{E_i\} \text{ paritions } E} \sum_{i\geq 1} |\mu(E_i)| \end{aligned}$$

So for any such λ we have

$$\lambda(E) \geq |\mu|(E)$$

$$|\mu|(E) \geq |\mu(E)|$$

But actually $|\mu|$ itself is a measure!

> **Definition.** The **total variation** of a complex measure $(X, \mathfrak{M}, \mu)$ is
>
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>
> is a positive measure on $(X, \mathfrak{M})$ and
>
> $$|\mu|(X) < \infty$$
>
> Therefore,
>
> $$|\;| : \mathrm{Meas}(\mathfrak{M}, \mathbb{C}) \to \mathrm{Meas}(\mathfrak{M}, (0, \infty))$$

So our problem of dominating μ with a positive measure does have a solution.

space of all complex measures

Definition. Normed space of all complex measures on $(X, \mathfrak{M})$

Let $(X, \mathfrak{M})$ be a measurable space then space of all complex measures on $\mathfrak{M}$

$$\mathrm{Meas}(\mathfrak{M}, \mathbb{C})$$

is a $\mathbb{C}$-vector space with norm being the variation measure of X

$$\|\mu\| := |\mu|(X)$$

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decomposition of signed measures

Definition. For a $\mathbb{R}$-measure μ we may decompose

$$\begin{aligned} \mu^+ &:= \frac{1}{2}(|\mu| + \mu) \\ \mu^- &:= \frac{1}{2}(|\mu| - \mu) \end{aligned}$$

which are positive measure called **positive** and **negative variations** of μ respectively.

Absolute continuity

Definition. Absolute continuity of measures

Let $\mu : \mathfrak{M} \to [0, \infty]$ be a positive measure on a measure space $(X, \mathfrak{M})$ and let

$$\lambda : \mathfrak{M} \to [0, \infty] \text{ or } \mathbb{C}$$

be an arbitrary measure (positive or complex). Then

- λ is **absolutely continuous** with respect to μ, $\lambda \ll \mu$ if

$$\forall E \in \mathfrak{M}, (\mu(E) = 0 \implies \lambda(E) = 0)$$

- λ is **concentrated** on $A \in \mathfrak{M}$ if

$$\begin{aligned} &\forall E \in \mathfrak{M}, \lambda(E) = \lambda(A \cap E) \\ \iff &\forall E \in \mathfrak{M}, E \cap A = \emptyset \implies \lambda(E) = 0 \end{aligned}$$

decomposition of complex measures

(Lebesgue decomposition of complex measure relative to a positive measure and Radon-Nikodym description of λ_a) Let $(X, \mathfrak{M}, \mu)$ be a positive σ-finite measure space and λ be a complex measure on $\mathfrak{M}$. Then there is a unique pair of complex measures λ_a, λ_s on $\mathfrak{M}$ such that

$$\begin{aligned} \lambda &= \lambda_a + \lambda_s \\ \lambda_a &\ll \mu \\ \lambda_s &\perp \mu \end{aligned}$$

called the **Lebesgue decomposition** of λ relative to μ.
If $\lambda : \mathfrak{M} \to [0, \infty)$ is positive and finite, then so are λ_a, λ_s.
Moreover there **exists an unique** $h \in L^1(X, \mathfrak{M}, \mu)$ such that

$$E \in \mathfrak{M} \implies \lambda_a(E) = \int_E h \, \mathrm{d}\mu$$

and is called the *Raydon-Nikodym derivative* of λ_a with respect to μ

$$\mathrm{d}\lambda_a = h\mathrm{d}\mu \text{ or } h = \frac{\mathrm{d}\lambda_a}{\mathrm{d}\mu}$$

consequences of the decomposition

(Polar decomposition of $\mathbb{C}$-measures) Let $(X, \mathfrak{M}, \mu)$ be a $\mathbb{C}$-measure space. Then there is a measurable function

$$h : X \to U(1) \subset \mathbb{C}$$

such that

$$\mathrm{d}\mu = h\mathrm{d}|\mu|$$

Current note has 0 direct children and 0 total descendants.

- structures based on sets
  - Measure spaces
    - $\mathbb{C}$-measures

And it has 7 siblings.

- structures based on sets
  - Measure spaces
    - Algebra of subsets of a set
    - Meas
    - Functions on measure spaces
    - $\mathbb{C}$-measures
    - $[0, \infty]$-measure spaces
    - Measure preserving endomorphisms
    - $[0, 1]$-measure spaces