> ⓘ Info
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> is written (completely with human hands) by Rupadarshi Ray,
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$\mathsf{FinVec}_{\mathbb{C}}$ irreducible representations of $\mathfrak{sl}(2,\mathbb{C})$

We want all representations

$$\underbrace{\mathfrak{sl}(2,\mathbb{C})}_{\cong_{\mathsf{Alg}}\mathbb{C}\left\langle H,X,Y \,\middle|\, \substack{[H,X]=2X\\ [H,Y]=-2Y\\ [X,Y]=H}\right\rangle} \to \mathrm{End}\mathsf{FinVec}_{\mathbb{C}}(V)$$

up to isomorphism.

- Start with an arbitrary finite dim irrep V. How does H, X, Y act on V?
  - The adjoint action by H

    $$H \overset{\mathrm{ad}}{\curvearrowright} \mathfrak{sl}(2,\mathbb{C})$$

    is diagonal. By *preservation of Jordan decomposition* under the representation

    $$\mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{gl}(V)$$

    the action $H \curvearrowright V$ must also be diagonal: hence we have a decomposition

    $$V = \bigoplus V_\alpha$$

    where V_α is the subspace such that

    $$v \in V_\alpha \implies H(v) = \alpha v$$

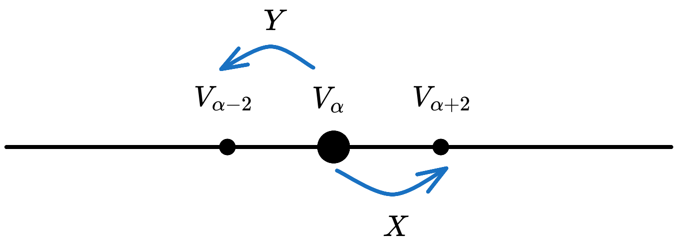
  - If $v \in V_\alpha$,

    $$\begin{aligned} H(X(v)) &= X(H(v)) + [H,X](v) \\ &= X(\alpha v) + 2X(v) \\ &= (\alpha+2)X(v) \end{aligned}$$

    Hence, $X : V_\alpha \to V_{\alpha+2}$.
  - If $v \in V_\alpha$,

    $$\begin{aligned} H(Y(v)) &= Y(H(v)) + [H,Y](v) \\ &= Y(\alpha v) - 2Y(v) \\ &= (\alpha-2)Y(v) \end{aligned}$$

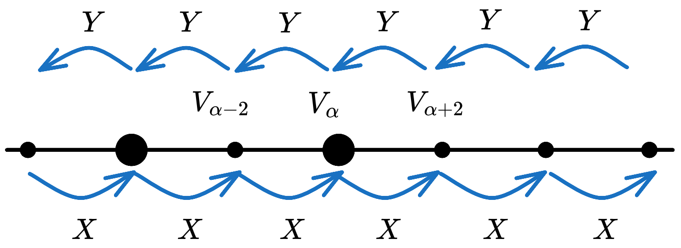
    Hence, $Y : V_\alpha \to V_{\alpha-2}$.
  - So,

- But

  $$\sum_{n\in\mathbb{Z}} V_{\alpha_0+2n}$$

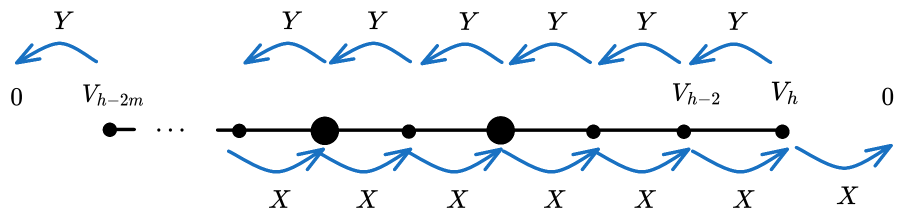
  is invariant under X, Y, H

  implies this subspace is invariant under the $\mathfrak{sl}_2\mathbb{C}$ action. So, irreducibility implies $V = \bigoplus V_{\alpha_0+2n}$.
  - Because V is finite dim, V_α that appear must satisfy

    $$\alpha \in \{h, h-2 \dots, h-2m\}$$

    for some $h \in \mathbb{C}$ ("highest" eigenvalue for H) and $m \in \mathbb{N}$ (the "size" of the representation) where the rest of the spaces are 0.
  - Thus we have

- Pick $0 \neq v \in V_n$ and then we get the subspace

  $$W := \mathbb{C}\{v, Y(v), \dots, Y^m(v)\} \leq V$$

  - Y preserves W by definition.
  - We know $Y^k(v) \in V_{h-2k}$, so H acts by

    $$H(Y^k(v)) = (h-2k)Y^k(v)$$

  - So H preserves W.
  - Now $X(v) = 0$ because v belongs to the space with last V_α.
    - **Base case:**

      $$\begin{aligned} X(Y(v)) &= [X,Y](v) + Y(X(v)) \\ &= H(v) + Y(0) \\ &= hv \end{aligned}$$

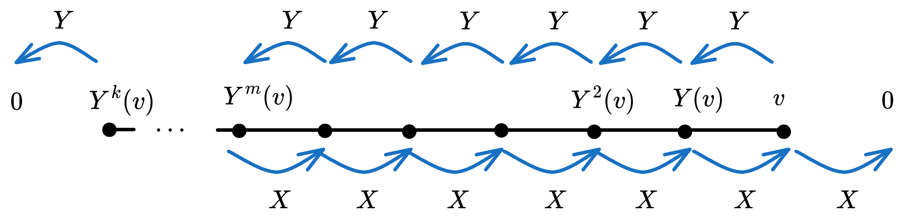
- **Induction hypothesis:** We assume

$$X(Y^k(v)) = k(h-k+1)Y^{k-1}(v)$$

- **Induction step:** We obtain

$$\begin{aligned} XY^{k+1}(v) = XY(Y^k(v)) &= [X,Y](Y^k v) + YXY^k v \\ &= HY^k(v) + Y(k(h-k+1)Y^{k-1}(v)) \\ &= (h-2m)Y^k(v) + k(h-k+1)Y^k(v) \\ &= h + k(h-k-1)Y^k(v) \\ &= (k+1)(h-k)Y^k(v) \end{aligned}$$

- Hence, $X(Y^k(v)) = k(h-k+1)Y^{k-1}(v)$ is true for all $k \geq 1$.

- So, X preserves W.
- 

- Hence, *by irreducibility*, we must have $V = W$. Hence, $\{v, Y(v), \dots, Y^k(v)\}$ is a basis of V.
  - $V_{n-2m} = \mathbb{C}Y^m(v)$ (one dimensional)
  - Because $Y^{m+1}(v) = 0$ we have

$$0 = XY^{m+1}(v) = (m+1)(h-m)Y^k(v)$$

  but $Y^k(v) \neq 0$ by definition, so

$$h - m = 0 \implies h = m$$

  and thus this implies the highest eigenvalue $h \in \mathbb{N}$ is a natural number and equals to the size of the representation.
  - Hence, for any such irreducible representation, we have a $n \in \mathbb{N}$ such that the eigenvalues of H are

$$\{\underbrace{n}_{h}, n-2, \dots, -n+2, -n\}$$

  and dimension of the representation is $n+1$. *This does now show existence, yet.*
- **Existence:** For each $n \in \mathbb{N}$ there exists a finite dim $\mathsf{Vec}_{\mathbb{C}}$ irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$:
  - $n = 0$ is the trivial representation
  - $n = 1$ the standard representation

$$\mathfrak{sl}(2,\mathbb{C}) \to \mathrm{EndVec}_{\mathbb{C}}(\underbrace{\mathbb{C}^2}_{\mathbb{C}\{x,y\}})$$

  where $Hx = x, Hy = -y$
  - and for any $n \in \mathbb{N}$ we have the representation

$$\mathfrak{sl}(2,\mathbb{C}) \to \mathrm{EndVec}_{\mathbb{C}}(\underbrace{Sym^n\mathbb{C}^2}_{\mathbb{C}\langle x^n, x^{n-1}y, \dots, y^n\rangle})$$

  such that $H(x^{n-k}y^k) = (n-2k)x^{n-k}y^k$

decomposing tensor product of irreps

geometric side

The automorphism group of $\mathbb{C}P^n$ - either as a algebraic variety or as a complex manifold - is the group

$$PGL_{\mathbb{C}}(n+1)$$

Correspondences

For any vector space W of dim $n+1$,

$$\boldsymbol{Sym}^k W^* \leftrightarrow \{\text{homog polynomials of deg } k \text{ on } \underbrace{\boldsymbol{P}(W)}_{= \text{space of lines in } W}\}$$

Dually,

$$\boldsymbol{Sym}^k W \leftrightarrow \{\text{homog polynomials of deg } k \text{ on } \underbrace{\boldsymbol{P}(W^*)}_{\substack{= \text{space of lines in } W^* \\ = \text{space of hyperplanes in } W}}\}$$

where the last correspondence is

$$\begin{aligned} \text{lines in } W^* &\leftrightarrow \text{hyperplanes in } W \\ [\phi] &\leftrightarrow \ker \phi \end{aligned}$$

Now because zero sets of polynomials are *hypersurfaces*(?)

$$P(\boldsymbol{Sym}^k W) \leftrightarrow \{\text{hypersurfaces of deg } k \text{ in } P(W^*)\}$$

so to derive results about $\boldsymbol{Sym}^k W$ we must work with $P(W^*)$.

Definition. Veronese embedding

For any vector space V we have the *Veronese embedding*

$$\begin{aligned} P(V^*) &\hookrightarrow P(\boldsymbol{Sym}^k V^*) \\ [v] &\mapsto [v^n] \end{aligned}$$

For $\dim(V) = 2$, $P(V^*) \cong \mathbb{C}P^1$, $\boldsymbol{Sym}^n V^*$ is of dim $n+1$, so it is $P(\boldsymbol{Sym}^n V^*) \cong \mathbb{C}P^n$ naturally and we have

$$\iota_n : \mathbb{C}P^1 \hookrightarrow \mathbb{C}P^n$$
$$x\hat{x} + y\hat{y} \mapsto (x\hat{x} + y\hat{y})^n$$
$$[x, y] \mapsto [x^n, x^{n-1}y, x^{n-2}y^2, \dots, xy^{n-1}, y^n]$$

whose image is called *rational normal curve of degree n*.

Now we have the representation

$$SL_{\mathbb{C}}(2) \curvearrowright \boldsymbol{Sym}^n V^* \cong \mathbb{C}^{n+1}$$
$$\hat{x}^k \hat{y}^{n-k} \overset{S}{\mapsto} (S\hat{x})^k (S\hat{y})^{n-k}$$

which quotients down to

$$SL_{\mathbb{C}}(2) \curvearrowright P\boldsymbol{Sym}^n V^* \cong \mathbb{C}P^n$$
$$\hat{x}^k \hat{y}^{n-k} \overset{S}{\mapsto} (S\hat{x})^k (S\hat{y})^{n-k}$$

the exact action on $\mathbb{C}P^n$ that preserves the *rational normal curve*.

weight character of a $\mathfrak{sl}(2, \mathbb{C})$ representation

Definition. Weight character of a $\mathfrak{sl}(2, \mathbb{C})$ representation

Because any $\mathbf{FinVec}_{\mathbb{C}}$ representation of $\mathfrak{sl}(2, \mathbb{C})$ is reducible, we may decompose it into its irreducible pieces

$$V \cong \bigoplus_{a \in \mathbb{N}} \boldsymbol{Sym}^a (\mathbb{C}^2)^{\otimes d_a}$$

and further, the irreps into its weight subspaces, that is, the eigenvalue of H which is brought all together as

$$V = \sum_{n \in \mathbb{Z}} \ker(H - n\mathsf{Id})$$

We define the **weight character** of this representation as the $\mathbb{Z}[t, t^{-1}]$ polynomial

$$\eta(V) := \sum_{n \in \mathbb{Z}} \dim \ker(H - n\mathsf{Id}_V) t^n$$

$$\eta(V \oplus W) = \eta(V) + \eta(W)$$

$$V \cong_{\mathfrak{sl}(2,\mathbb{C})} W \iff \eta(V) = \eta(W)$$

We know the weight character of the irreducible reps:

$$\eta(\boldsymbol{Sym}^n(\mathbb{C}^2)) = t^{-n} + t^{-n+2} + \dots + t^{n-2} + t^n$$
$$= \frac{t^{n+2} - t^{-n}}{t^2 - 1}$$

But the main use of the ring of polynomials is here:

$$\eta(V \otimes W) = \eta(V)\eta(W)$$

which Laurent polynomials are achieved by representations?

$$\sum_{n \in \mathbb{N}} a_n (t^{-n} + t^{-n+2} + \dots + t^{n-2} + t^n)$$
$$= \sum_{i \in \mathbb{N}} \sum_j a_j (t^{-i} + t^i)$$

decompositions of tensor powers of irreps into sums of irreps

- $2 \otimes 3 = 5 \oplus 3 \oplus 1$

The tensor product

$\boldsymbol{Sym}^a(\mathbb{C}^2) \otimes$ S	$-a$	$-a+2$	$\dots$	$a-2$	a
$-b$	$-a-b$	$-a-b+2$		$a-b-2$	$a-b$
$-b+2$	$-a-b+2$	$-a-b+4$		$a-b$	$a-b+2$
$\dots$			$\dots$		
$b-2$	$b-a-2$	$b-a$		$a+b-4$	$a+b-2$
b	$b-a$	$b-a+2$		$a+b-2$	$a+b$

gives the weight decomposition from top-left to bottom right

	$-a-b$	$-a-b+2$	
	1	2	

$$\boldsymbol{Sym}^a(\mathbb{C}^2) \otimes \boldsymbol{Sym}^b(\mathbb{C}^2) \cong_{SL_{\mathbb{C}}(2)} \bigoplus_{i=0}^{b} \boldsymbol{Sym}^{a+b-2i}(\mathbb{C}^2)$$

decompositions of symmetric powers of irreps into sums of irreps

- $\boldsymbol{Sym}^2 \boldsymbol{Sym}^2(\mathbb{C}^2) \cong_{SL_{\mathbb{C}}(2)} \boldsymbol{Sym}^0(\mathbb{C}^2) \oplus \boldsymbol{Sym}^4(\mathbb{C}^2)$
-

Current note has 2 direct children and 2 total descendants.

- stem
  - SL C 2
    - rep
      - $\mathrm{FinVec}_{\mathbb{C}}$ irreducible representations of $\mathfrak{sl}(2, \mathbb{C})$
        - $2 \otimes 3 = 5 \oplus 3 \oplus 1$ for $\mathrm{FinVec}_{\mathbb{C}}$ irreps of $\mathfrak{sl}(2, \mathbb{C})$
        - $\boldsymbol{Sym}^2 \boldsymbol{Sym}^2(\mathbb{C}^2) \cong_{SL_{\mathbb{C}}(2)} \boldsymbol{Sym}^0(\mathbb{C}^2) \oplus \boldsymbol{Sym}^4(\mathbb{C}^2)$

And it has 1 siblings.

- stem
  - SL C 2
    - rep
      - $\mathrm{FinVec}_{\mathbb{C}}$ irreducible representations of $\mathfrak{sl}(2, \mathbb{C})$