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Hardy-Littlewood maximal functions of $L^1_{\text{loc}}(\mathbb{R}^n)$

Definition. Hardy-Littlewood maximal function of $L^1_{\text{loc}}(\mathbb{R}^n)$

Let $f \in L^1_{\text{loc}}(\mathbb{R}^n, m)$ then

•
$$(Mf)(x) := \sup_{\delta > 0} \frac{1}{m(B(x, \delta))} \int_{B(x, \delta)} |f|$$

is the **centered Hardy-Littlewood maximal function**

•
$$(Mu f)(x) := \sup_{\delta > 0, |x-y| < \delta} \frac{1}{m(B(y, \delta))} \int_{B(y, \delta)} |f|$$

is the **uncentered** one

•
$$M(f) = M(|f|) \geq 0$$

and M is thus positive and semi-linear

•
$$M : L^\infty \rightarrow L^\infty$$

with

$$\|M(f)\|_\infty \leq \|f\|_\infty$$

•
$$M(f) \leq Mu(f) \leq 2^n M(f)$$

• The linear functional

$$\begin{aligned} (Mu f)(x) &= \sup_{\delta > 0} \frac{1}{m(B_1)\delta^n} \int_{y \in \mathbb{R}^n} |f(x-y)| \chi_{B_1}\left(\frac{y}{\delta}\right) \\ &= \sup_{\delta > 0} |f| * \underbrace{\frac{\chi_{B_1}\left(\frac{y}{\delta}\right)}{\delta^n}}_{\chi_{B_1, \delta}} \end{aligned}$$

is thus supremum of convolutions with dilations of the characteristic of the ball.

maximal inequalities

Lemma 2.1.5. *Let $\{B_1, B_2, \dots, B_k\}$ be a finite collection of open balls in $\mathbf{R}^n$. Then there exists a finite subcollection $\{B_{j_1}, \dots, B_{j_l}\}$ of pairwise disjoint balls such that*

$$\sum_{r=1}^l |B_{j_r}| \geq 3^{-n} \left| \bigcup_{i=1}^k B_i \right|. \tag{2.1.2}$$

Proof. Let us reindex the balls so that

$$|B_1| \geq |B_2| \geq \dots \geq |B_k|.$$

Let $j_1 = 1$. Having chosen $j_1, j_2, \dots, j_i$, let j_{i+1} be the least index $s > j_i$ such that $\bigcup_{m=1}^i B_{j_m}$ is disjoint from B_s . Since we have a finite number of balls, this process will terminate, say after l steps. We have now selected pairwise disjoint balls $B_{j_1}, \dots, B_{j_l}$. If some B_m was not selected, that is, $m \notin \{j_1, \dots, j_l\}$, then B_m must intersect a selected ball B_{j_r} for some $j_r < m$. Then B_m has smaller size than B_{j_r} and we must have $B_m \subseteq 3B_{j_r}$. This shows that the union of the unselected balls is contained in the union of the triples of the selected balls. Therefore, the union of all balls is contained in the union of the triples of the selected balls. Thus

$$\left| \bigcup_{i=1}^k B_i \right| \leq \left| \bigcup_{r=1}^l 3B_{j_r} \right| \leq \sum_{r=1}^l |3B_{j_r}| = 3^n \sum_{r=1}^l |B_{j_r}|,$$

and the required conclusion follows. □

(Hardy–Littlewood maximal inequality) The operators

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map

$$M, Mu : L^1 \rightarrow L^{1, \infty}$$

with norm $\leq 3^n$ and

$$M, Mu : L^p \rightarrow L^p$$

with norm $\leq 3^{n/p}\frac{p}{p-1}$. Moreover for $f \in L^1, \alpha > 0$ we have weak-boundedness

$$m(Mf > \alpha) \leq \frac{c(n)}{\alpha} \int_{Mf > \alpha} |f|$$

where $c(n) \leq 3^n$

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