Info

This note, that was found here, as a part of a collection
is written (completely with human hands) by Rupadarshi Ray,
was created on May 26, 2026 1:26:33 PM,
was last modified on September 7, 2026 1:10:55 PM, and
was exported on September 7, 2026 3:09:52 PM.

Dyadic cubes

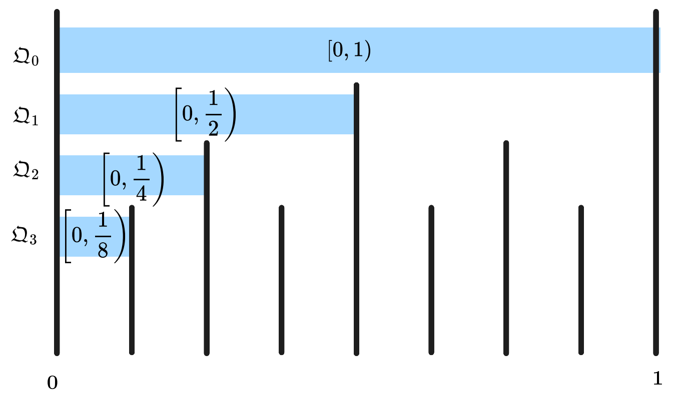
Definition. Dyadic cubes

A **dyadic cube** in $\mathbb{R}^d$ of *generation* $n \in \mathbb{Z}$ is an element of

$$\mathfrak{D}_n := \left\{ 2^n (k + [0,1)^d) \middle| k \in \mathbb{Z}^d \right\}$$

$d = 1$

$$\mathfrak{D}_0 = \{\dots, [-1,0), [0,1), [1,2), [2,3), \dots\}$$
$$\mathfrak{D}_{-1} = \left\{\dots, \frac{1}{2}[-1,0), \frac{1}{2}[0,1), \frac{1}{2}[1,2), \frac{1}{2}[2,3), \dots\right\}$$



Therefore, dyadic cubes of generation $-m$, $\mathfrak{D}_{-m}$, for $m \geq 1$ partitions $0 \in \mathfrak{D}_0$ into 2^m subsets

$$\left[0, \frac{1}{2^m}\right), \left[\frac{1}{2^m}, \frac{2}{2^m}\right), \dots, \left[\frac{2^m - 1}{2^m}, 1\right)$$

partition using dyadic cubes

> **Proposition:**
>
> $$\mathbb{R}^d = \bigsqcup \mathfrak{D}_n$$

> **Proposition:** Let Q_1 and Q_2 be two dyadic cubes that intersect $Q_1 \cap Q_2 \neq \emptyset$. Then either $Q_1 \subseteq Q_2$ or $Q_2 \subseteq Q_1$.

covering lemma

(Covering lemma for dyadic cubes) Let $Q_1, \dots, Q_N$ be a finite collection of dyadic cubes. Then there is a subcollection $Q_{n_1}, \dots, Q_{n_M}$ of **disjoint** dyadic cubes such that

$$Q_{n_1} \sqcup \dots \sqcup Q_{n_M} = Q_1 \cup \dots \cup Q_N$$

Consider the maximal dyadic cubes in the collection

$$\{Q_{n_1}, \dots, Q_{n_M}\} := \max(\{Q_1, \dots, Q_N\}, \subseteq)$$

- By

  > **Proposition:** Let Q_1 and Q_2 be two dyadic cubes that intersect $Q_1 \cap Q_2 \neq \emptyset$. Then either $Q_1 \subseteq Q_2$ or $Q_2 \subseteq Q_1$.

  either Q_{n_i} are all disjoint or they are not maximal.

σ-algebra

Definition. Let

$$\langle \mathfrak{D}_n \rangle$$

be the σ-algebra generated by dyadic cubes of generation n.

> **Proposition:**
>
> $$\langle \mathfrak{D}_n \rangle = \{\text{countable union of elements in } \mathfrak{D}_n\}$$

> **Proposition:** A function $f : \mathbb{R}^d \to \mathbb{C}$ is $\langle \mathfrak{D}_n \rangle$-measurable $\iff$ f is constant on each dyadic cube in $\mathfrak{D}_n$.

Because

$$\langle \mathfrak{D}_n \rangle \subset \mathfrak{B}_{\mathbb{R}^d}$$

we have

$$L^2(\mathbb{R}^d, \langle \mathfrak{D}_n \rangle, m) \hookrightarrow L^2(\mathbb{R}^d, \mathfrak{B}_{\mathbb{R}^d}, m)$$

is an isometric embedding. This is of course simply the inclusion of functions constant on each dyadic cube of generation n.

Its adjoint, the conditional expectation is averaging over each dyadic cube:

$$\boldsymbol{E}(f | \langle \mathfrak{D}_n \rangle)(x) = \left\{ \frac{1}{m(Q)} \int_Q f \quad x \in Q \in \mathfrak{D}_n \right.$$

Current note has 0 direct children and 0 total descendants.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $\Gamma\backslash\mathbb{R}^n, \Gamma\backslash\mathbb{C}^n$
    - Dyadic cubes

And it has 39 siblings.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $\Gamma\backslash\mathbb{R}^n, \Gamma\backslash\mathbb{C}^n$
    - Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
    - Open subsets of $\mathbb{R}^n$ with $\mathcal{C}^2$ boundary
    - Complex ODEs in $\mathbb{C}$
    - Circle packing on $\mathbb{R}^2$
    - Circle packing converges to the Riemann biholomorphism
    - Bounded connected open subsets of $\mathbb{C}^n$
    - Connected circular open subsets of $\mathbb{C}^n$
    - Continuous functions on $\mathbb{R}^d$
    - Dyadic cubes
    - Curves
    - Differentiable functions
    - Differential forms on $\mathbb{R}^n$
    - Fourier-Wigner transform
    - Harmonic functions composed with conformal maps
    - Hilbert transform
    - Fourier-Cauchy-Poisson correspondence of holomorphic and harmonic functions on the unit disk and their boundary values
    - Holomorphic subsets of $\mathbb{C}^n$
    - Regular hypersurfaces in $\mathbb{R}^{2n}$
    - Orientable hypersurfaces in $\mathbb{R}^n$
    - Laplace operator on $\mathbb{R}^n$
    - $l^\infty(\mathbb{R})$
    - Lebesgue measurable subsets of and functions on $\mathbb{R}^n, T^n, S^n$
    - Lebesgue measure of boundary of open sets in $\mathbb{R}^n$
    - Local $\mathbb{R}$-algebras
    - $l^p(\mathbb{R})$
    - Metric density of subsets of $\mathbb{R}^n$
    - Monotone functions on $\mathbb{R}$
    - Cauchy integral of periodic functions
    - Polygons with integer vertices
    - Open smooth maps $U \subseteq \mathbb{R}^2 \to \mathbb{C}$
    - Discrete subgroups of $\mathbb{R}^n$
    - Discrete cocompact subgroups of $\mathbb{R}^n$, flat tori
    - Ramified germs of smooth and holomorphic functions
    - Open subsets of $\mathbb{R}^n$
    - Open subsets of $\mathbb{R}^n$ equipped with the flat metric
    - Closed subsets of $\mathbb{R}^n$ equipped with smooth functions
    - Quasi-analytic smooth functions on $\mathbb{R}$
    - Star-shaped subsets of $\mathbb{R}^n$
    - ODEs in $\mathbb{R}^n \leftrightarrow$ Vector fields in $\mathbb{R}^n$