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Integration by parts

- Let $f, g \in \mathcal{C}^1[a, b]$ then

$$(fg)' = f'g + g'f$$

and

Let f be Riemann-Darboux integrable on $[a, b]$ and let F be differentiable on $[a, b]$ such that $F' \equiv f$. Then

$$\int_{[a,b]} f = F(b) - F(a)$$

implies

$$f(b)g(b) - f(a)g(a) = \int_{[a,b]} (f'g + fg')$$

We may assume f, g to be absolutely continuous.

(Fundamental theorem of analysis, in the almost everywhere sense) A function

$$F : [a, b] \rightarrow \mathbb{R}$$

is *absolutely continuous* on $[a, b] \implies F'$ exists m -almost everywhere and is L^1 . In that case,

$$F(x) = F(a) + \int_{[a,x]} F' \quad \text{on } [a, b]$$

Moreover $F' = 0$ m -almost everywhere $\implies F$ is constant m -almost everywhere. Conversely, for every $f \in L^1[a, b]$ there exists a function F which is differentiable m -almost everywhere and

$$F' = f \text{ ae}$$

and in fact we may take F to be the absolutely continuous function $x \mapsto c_0 + \int_{[a,x]} f$ for any $c_0 \in \mathbb{C}$.

for absolutely continuous functions

However, we do not need f, g to be *differentiable* for the integration by parts to be true.

- Let

$$f, g : [a, b] \rightarrow \mathbb{C}$$

be such that there exists $\phi, \psi \in L^1[a, b]$ satisfying

$$\begin{aligned} f(x) &= c_1 + \int_{[a,x]} \phi \\ g(x) &= c_2 + \int_{[a,x]} \psi \end{aligned}$$

for $c_1, c_2 \in \mathbb{R}$. In particular, $f, g \in \mathcal{C}[a, b]$.

- As $\phi, \psi \in L^1[a, b]$ we have

$$(x, y) \mapsto \phi(x)\psi(y)$$

in $L^1[a, b]^2$.

- By

(Fubini's theorem) Suppose $f \in L^1(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$. Then for almost every $y \in \mathbb{R}^{d_2}$

$$f(-, y) \in L^1(\mathbb{R}^{d_1})$$

and

$$y \mapsto \int_{\mathbb{R}^{d_1}} f(-, y) \in L(\mathbb{R}^{d_2})$$

Moreover,

$$\int_{y \in \mathbb{R}^{d_2}} \left(\int_{x \in \mathbb{R}^{d_1}} f(x, y) \right) = \int_{\mathbb{R}^{d_1+d_2}} f$$

$$\begin{aligned} \int_{[a,b]} (f\phi + g\psi) &= \int_{t \in [a,b]} g(t) \int_{s \in [a,t]} f(s) + \int_{t \in [a,b]} f(t) \int_{s \in [a,t]} g(s) \\ &= \int_{[a,b] \times [a,b]} f \otimes g \\ &= \left(\int_{[a,b]} f \right) \left(\int_{[a,b]} g \right) \\ &= (f(b) - f(a))(g(b) - g(a)) \end{aligned}$$

when $f'g$ and fg' are not L^1

[1]

- subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $\Gamma \backslash \mathbb{R}^n, \Gamma \backslash \mathbb{C}^n$
 - int
 - Integration by parts

And it has 4 siblings.

- stem
 - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $\Gamma \backslash \mathbb{R}^n, \Gamma \backslash \mathbb{C}^n$
 - int
 - Integration by parts
 - Laplace's method for approximating integral of e^{Mf} for large M
 - Integral on bounded open subsets of $\mathbb{R}^n$ with $\mathcal{C}^1$ boundary
 - Riemann-Darboux and Riemann-Stieltjes integrals

1. <https://faculty.uml.edu/jpropp/parts.pdf> ↩