Info

This note, that was found here, as a part of a collection
is written (completely with human hands) by Rupadarshi Ray,
was created on May 6, 2024 1:27:50 AM,
was last modified on September 7, 2026 2:27:27 PM, and
was exported on September 7, 2026 3:09:57 PM.

Complex ODEs in $\mathbb{C}$

Complex ODEs of order 2 in $\mathbb{C}$

	we may convert the Cauchy problem	
	$y^{(n)}(z)+\sum_{i=0}^{n-1} b_i(z)y^i(z)=0$	$\left[\frac{\partial w_i}{\partial z}\right]=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \dots & & \\ -b_1 & -b_2 & -b_3 & . \end{bmatrix}$
solution	$y(z)$	$w(z)\in\mathscr{O}(X,\mathbb{C}^n)$
global solution on simply connected X		
onto the simply connected cover $p:U(X)\to X$		$\mathrm{d}w=(p^*A)w$

matrix differential equations on simply connected domains

For any holomorphic

$$A:U\to \mathsf{Mat}_{\mathbb{C}}(n)$$

on a **simply connected** open $U\subseteq\mathbb{C}$ and any choices of $z_0\in U$ and $w_0\in\mathbb{C}^n$, there exists an unique holomorphic function $w:U\to\mathbb{C}^n$ such that

$$\frac{\partial}{\partial z}w(z)=A(z)w(z)\text{ on }U$$

such that $w(z_0)=w_0$.

on $\mathbb{C}P^1\setminus\{a_1,\dots,a_n\}$

Let $X:=\mathbb{C}P^1\setminus\{a_1,\dots,a_n\}$ and

$$p:U(X)\to X$$

be the simply connected cover.

The bundles are trivial, so we need not worry about sections. (?) We can easily translate between $\mathbb{C}^n$-valued forms and $\mathbb{C}^n$-valued functions

	$A:U\to\mathsf{Mat}_{\mathbb{C}}(n)$	$A\in\Omega^1\mathscr{O}(X,\mathsf{Mat}_{\mathbb{C}}(n))$
equation	$\frac{\partial}{\partial z}w(z)=A(z)w(z)$	$\mathrm{d}w=Aw$
solution	$\omega:U\to\mathbb{C}^n$	$w\in\Omega^1\mathscr{O}(X,\mathbb{C}^n)$
lift to universal over		$\mathrm{d}w=(p^*A)w$
fundamental systems	$\frac{\partial}{\partial z}S_A=AS_A$	$\mathrm{d}S_A=AS_A$

we lift the problem to the universal cover

Given $A\in\Omega^1\mathscr{O}(X,\mathsf{Mat}_{\mathbb{C}}(n))$, let

$$\mathcal{L}_A\le\mathscr{O}(U(X),\mathsf{Mat}_{\mathbb{C}}(n))$$

be the n-dimensional complex vector space of solutions to

$$\mathrm{d}w=(p^*A)w$$

Then

$$\begin{aligned} \pi_1(X)&\curvearrowright\mathcal{L}_A\\ w\overset{\sigma}{\mapsto}(\sigma^{-1})^*w&=w\circ\sigma^{-1}\end{aligned}$$

is a representation of $\pi_1(X)$.

fundamental system of solution

We may think of

$$S_A:U(X)\to GL_{\mathbb{C}}(n)$$

as the matrix formed by an ordered basis for $\mathcal{L}_A$.

action of π_1 on fundamental system

fundamental systems

	choice of "initial" or a basepoint	**choice of basis of solutions**
choice	$z_0\in U$	ordered basis $w_i\in\mathcal{L}_A$

	choice of "initial" or a basepoint	choice of basis of solutions
a fundamental system	For a given $A : U \to \mathsf{Mat}_{\mathbb{C}}(n)$ and $z_0 \in U$ $$S_{A,z_0} := [S_1 \quad S_2 \quad \dots \quad S_n] : U$$ be the holomorphic function such that the columns S_i is the unique solution of $$\frac{\partial S_i}{\partial z} = AS_i,\ S_i(z_0) = e_i$$	(w_i)
	$$U \to \mathrm{IsomVec}(\mathcal{L}_A, \mathbb{C}^n),\ z_0 \mapsto$$ chooses a basis for us	
	Then S_{A,z_0} uniquely satisfies $$\frac{\partial S_{A,z_0}}{\partial z} = AS_{A,z_0},\ S_{A,z_0}(z_0) =$$	
change of choice		
	Evidently $$w(z) = S_{A,z_0}(z)w_0$$ is the solution for the initial condition $w(z_0) = w_0$.	
Let $$\mathcal{F}_A \subseteq (\mathscr{O}(U, GL_{\mathbb{C}}(n)), \times)$$ be the set of all fundamental systems for all such *choices*.		What we achieved is: given a basis of $\mathcal{L}_A$, we constructed a fundamental system $$\mathrm{IsomVec}(\mathcal{L}_A, \mathbb{C}^n) \to \mathcal{F}_A \subseteq (\mathscr{O}$$ which *should be* a $GL_{\mathbb{C}}(n)$-torsor isomorphism.
	$$\Delta_{A,z_0} := \det S_{A,z_0}$$ uniquely solves the differential equation $$\frac{\partial}{\partial z}\Delta_{A,z_0} = \mathrm{tr}(A)\Delta_{A,z_0}$$	
dependence on A	We package this in the in the description $$\mathscr{O}(U, \mathsf{Mat}_{\mathbb{C}}(n)) \times U \to (\mathscr{O}(U,$$ $$(A, z_0) \mapsto S_{A,z_0}$$ which speaks only of $U \subseteq \mathbb{C}$ and matrices and nothing of $\mathbb{C}^n$, but one may think of this as the "exponential" of the holomorphic matrix vector field on $\mathbb{C}^n$?	$$A \mapsto \mathcal{F}_A$$

> Claim:
>
> $$S_A : U \to GL_{\mathbb{C}}(n)$$

Current note has 1 direct children and 1 total descendants.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $_\Gamma\backslash^{\mathbb{R}^n}, _\Gamma\backslash^{\mathbb{C}^n}$
    - Complex ODEs in $\mathbb{C}$
      - Complex ODEs of order 2 in $\mathbb{C}$

And it has 39 siblings.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $_\Gamma\backslash^{\mathbb{R}^n}, _\Gamma\backslash^{\mathbb{C}^n}$
    - Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
    - Open subsets of $\mathbb{R}^n$ with $\mathcal{C}^2$ boundary
    - Complex ODEs in $\mathbb{C}$
    - Circle packing on $\mathbb{R}^2$
    - Circle packing converges to the Riemann biholomorphism
    - Bounded connected open subsets of $\mathbb{C}^n$
    - Connected circular open subsets of $\mathbb{C}^n$
    - Continuous functions on $\mathbb{R}^d$
    - Dyadic cubes
    - Curves
    - Differentiable functions
    - Differential forms on $\mathbb{R}^n$
    - Fourier-Wigner transform
    - Harmonic functions composed with conformal maps
    - Hilbert transform
    - Fourier-Cauchy-Poisson correspondence of holomorphic and harmonic functions on the unit disk and their boundary values
    - Holomorphic subsets of $\mathbb{C}^n$
    - Regular hypersurfaces in $\mathbb{R}^{2n}$
    - Orientable hypersurfaces in $\mathbb{R}^n$

- Laplace operator on $\mathbb{R}^n$
- $l^\infty(\mathbb{R})$
- Lebesgue measurable subsets of and functions on $\mathbb{R}^n, T^n, S^n$
- Lebesgue measure of boundary of open sets in $\mathbb{R}^n$
- Local $\mathbb{R}$-algebras
- $l^p(\mathbb{R})$
- Metric density of subsets of $\mathbb{R}^n$
- Monotone functions on $\mathbb{R}$
- Cauchy integral of periodic functions
- Polygons with integer vertices
- Open smooth maps $U \subseteq \mathbb{R}^2 \to \mathbb{C}$
- Discrete subgroups of $\mathbb{R}^n$
- Discrete cocompact subgroups of $\mathbb{R}^n$, flat tori
- Ramified germs of smooth and holomorphic functions
- Open subsets of $\mathbb{R}^n$
- Open subsets of $\mathbb{R}^n$ equipped with the flat metric
- Closed subsets of $\mathbb{R}^n$ equipped with smooth functions
- Quasi-analytic smooth functions on $\mathbb{R}$
- Star-shaped subsets of $\mathbb{R}^n$
- ODEs in $\mathbb{R}^n \leftrightarrow$ Vector fields in $\mathbb{R}^n$