Summer School on Rigidity of Discrete Groups, June 30 – July 4, 2025 at IISER Mohali

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Organisers:

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In the proposed meeting, we shall focus on some aspects of the broad topic Rigidity in Geometry and Topology. The geometric form of Mostow's celebrated rigidity theorem states that every isomorphism between fundamental groups of closed hyperbolic manifolds of dimension at least 3, is induced by an isometry. This result was followed by a similar theorem of Prasad for nonuniform lattices. However, in a spectacular breakthrough, G. A. Margulis obtained a far reaching generalization of these theorems for irreducible lattices in higher rank groups. This result is now known as the Margulis' Superrigidity Theorem. The ideas behind proofs of these theorems have become very important and fruitful in modern mathematics. In the proposed workshop, we aim to provide a gentle introduction to these results along with similar rigidity theorem in geometry and topology through mini courses and survey talks.

***Speakers*:** Marc Bourdon (U. Lille), Pralay Chatterjee (IMSc), Soma Maity (IISER Mohali), Arghya Mondal (IISER Mohali), C. S. Rajan (Ashoka), Pranab Sardar (IISER Mohali), Riddhi Shah (JNU), T. N. Venkataramana (ICTS Bengaluru).

Expected Participants: Faculty members, PhD students and postdoctoral fellows working in related areas. The topics to be covered in this school require some prior knowledge of geometric group theory, smooth manifolds and Lie groups.

Due to limited funding, we shall be able to provide local hospitality to only a limited number of participants who will be selected based on the proximity of their research interests with the theme of the summer school. Participants are requested to arrange their own travel funding.

Name of the Speaker	No. of Lectures	Abstract
Marc Bourdon (MB), Université de Lille, France	4 lectures	Title: Lie groups and quasi-isometries Abstract: In geometric group theory, one studies groups as geometric objects, and tries to determine whether an algebraic property is a geometric one. In this setting, the natural maps between groups are the so-called quasi-isometries (QI). A class of groups is said to be QI-rigid if every finitely generated group that is QI to a group in the class, is virtually isomorphic to (another) group in the class. Geometric group theory started in the 80's when Gromov proved that the class of nilpotent groups is QI-rigid. In the 90's the subject focused on semisimple groups; it culminated when a combination of works allowed to establish that the class of lattices in a given semisimple group is QI-rigid. Since then, geometric group theory has developed several interactions with other fields of mathematics. It has been used to solve some old open problems (e.g. in 3-manifold topology). A major problem in geometric group theory is to classify connected Lie groups up to QI. One knows that every simply connected Lie group is QI to a completely solvable Lie group, i.e. to a closed subgroup of the upper triangular real matrix group. In 2018 Cornulier conjectured that two QI completely solvable groups must be isomorphic. This is currently open, even in the smaller class of nilpotent Lie groups. During the talks, I plan to introduce some QI-invariants for Lie groups, and illustrate them with examples. These invariants include the growth rate, the rank, and the

Name of the Speaker	No. of Lectures	Abstract
		L^p-cohomology. The examples of groups that will be discussed include the nilpotent groups, the Heintze groups, the abelian-by-abelian solvable Lie groups. Some known results about their QI classification will also be presented.
Pralay Chatterjee (PC), IMSc Chennai	3 lectures	Title: Introduction to Lattices Abstract: Semisimple Lie groups, Lattices in Lie groups, Some generalities on lattices in Lie groups, SL_n (Z) is a lattice in SL_n (R), Mahler's compactness criteria, Borel density theorem, Arithmetic lattices, Explaining the content of Borel-Harish-Chandra theorem and Margulis Arithmeticity theorem, Godment criteria for co-compactness of arithmetic lattices, Methods to produce arithmetic lattices.
T. N. Venkataramana (TNV) ICTS, Bengaluru	5 lectures	Title: A Proof of the Margulis Superrigidity Theorem Abstract: We shall review the proof of the Margulis superrigidity theorem following the paper by Margulis, Invent. Math. 1984, developing the necessary background. A brief outline is given below. Rank of a semisimple Lie group, Irreducible lattices, Uniform & non-uniform Lattices, Rank 1 vs Higher rank, Examples, Arithmeticity theorem with proof, Zariski dense subgroups, Furstenberg measure, Amenable subgroups, Existence of equivariant measurable maps, Induced representations, Rationality of equivariant measurable maps. Proof of the superrigidity theorem.

Name of the Speaker	No. of Lectures	Abstract
C. S. Rajan (CSR), Ashoka University	3 Lectures	Title: Harmonic superrigidity Abstract: The lectures will serve as an introduction to the rigidity results of Corlette and Gromov-Schoen.
Pranab Sardar (PS), IISER Mohali	1 lecture	Title: On some rigidity theorems in geometric group theory Abstract: The rigidity theorems of Mostow, Margulis and others for lattices in semisimple Lie groups inspired many such rigidity type results in geometry and topology. In this expository lecture, we will survey some such theorems related to geometric group theory.
Arghya Mondal	2 lectures	Title: Mostow rigidity Abstract: The Mostow Prasad Rigidity Theorem says if two finite volume hyperbolic manifolds of dimension greater than or equal to 3 are homotopy equivalent then they are isometric. We will give a complete proof of this statement for closed hyperbolic 3-manifolds. Then we will indicate how the proof can be extended in the general case.

Name of the Speaker	No. of Lectures	Abstract
Riddhi Shah (RS)	1 lecture	Title: The structure of automorphism groups and lattices in Lie groups Abstract: We study the structure of groups of automorphisms of a connected Lie group; namely, we identify certain conditions under which they are almost algebraic, and discuss some applications and examples (joint work with S.G. Dani - https://arxiv.org/abs/2504.18641). We explore the structure of lattices in a connected Lie group and discuss some properties of automorphisms which keep a lattice invariant (joint work with Rajdip Palit and Manoj B. Prajapati - Geometry, and Dynamics 17 (2023), 185-213 - https://doi.org/10.4171/GGD/672)

references

- Dave Witte Morris - Arithmetic groups
- Venkataramana-rigidity-notes.pdf
- Raghunathan - Discrete subgroups of Lie groups
- Marc Bourdon (Université de Lille): "Quasi-isometric inv of continuous group L^p cohomology & apps." Marc Bourdon - Quasi-isometric invariants of continuous group Lpp cohomology and applications
- Robert J. Zimmer - Ergodic Theory and Semisimple Groups (1984).pdf
- Werner Ballmann, Mikhael Gromov, Viktor Schroeder - Manifolds of Nonpositive Curvature (1985).pdf

What is a rigidity theorem?

Mostow rigidity theorem

Quasi-isometric rigidity

Definition. quasi-isometric embedding

Let X, Y are metric spaces and $k \geq 0$. Then

$$f : X \to Y$$

is called a **k-quasi isometric embedding** if

$$-k + \frac{1}{k} d(x, x') \leq d(f(x), f(x')) \leq k + d(x, x')$$

for all $x, x' \in X$.

Examples:

- $$\begin{aligned} f: \mathbb{R} &\to \mathbb{R}^2 \\ t &\mapsto (t, \cos t) \end{aligned}$$

  is a quasi embedd
- $$t \mapsto (t, t^2)$$

  is **not** a quasi embedding
- $$t \mapsto (t, |t|)$$

  is a quasi-isometry onto image

> $$\mathbb{R}^m \cong_{\text{quasi}} \mathbb{R}^n \implies n = m$$

> $$H^n \cong_{\text{quasi}} H^m \implies n = m$$

All our spaces are either manifolds... or graphs!

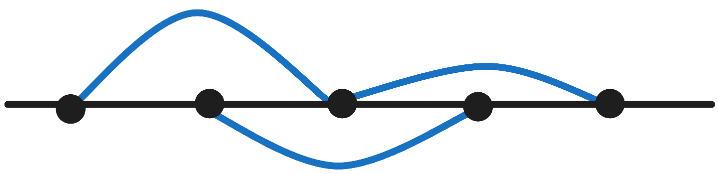
Consider a connected graph X and assign length 1 to each edge, which makes X a metric space

$$d(x, y) := \inf\{\text{length of paths from } x \text{ to } y\}$$

Now given a group Γ and $S \subset \Gamma$ we may consider its Caley graph $\mathrm{Cay}(\Gamma, S)$ with vertices Γ and edges on (g, gs) for all $g \in \Gamma$ and $s \in S$.

> **Exercise**
>
> $\mathrm{Cay}(\Gamma, S)$ is connected if $\Gamma = \langle S \rangle$.

Consider $\Gamma = \mathbb{Z}$ and $S = \{1, 2\}$ then the Caley graph is



> **(Milnor-Schwarz) Let X be a length metric space and $\Gamma \curvearrowright X$ acts by isometries properly such that X/Γ is compact then Γ is finitely generated and**
>
> $$\begin{aligned} \Gamma &\to X \\ g &\mapsto gx \end{aligned}$$
>
> **is a quasi-isometry**

Corollaries:

- $\Gamma_1 < \Gamma$ (f.g.) and Γ / Γ_1 is finite then

  $$\Gamma_1 \hookrightarrow \Gamma$$

is a quasi-isometry

- Let Γ be f.g. and $F \trianglelefteq \Gamma$ with $|F| < \infty$ then

$$\Gamma \to \Gamma/F$$

is a quasi-isometry.

- $$\mathrm{Cay}(\Gamma, S_1) \cong_{\text{quasi}} \mathrm{Cay}(\Gamma, S_2)$$

if $\langle S_1 \rangle = \langle S_2 \rangle = \Gamma$

Let X is a Riemannian symmetric space and $\Gamma < \mathrm{Isom}(X)$ is a uniform lattice. Then

$$\Gamma \to X$$
$$g \mapsto gx$$

is a quasi isometry.

Examples:

- $\mathbb{Z}^n \curvearrowright \mathbb{R}^n$
- $\pi_1(S_g) \curvearrowright H^2, g \geq 2$ then $\Gamma \cong_{\text{quasi}} H^2$

Definition. **Virtual isomorphism of groups**

Suppose Γ_1, Γ_2 are two f.g. groups. We say that Γ_1, Γ_2 are **virtually isomorphic** if there exists $\Gamma_1' < \Gamma_1, \Gamma_2' < \Gamma_2$ finite index and $F_1 \trianglelefteq \Gamma_1', F_2 \trianglelefteq \Gamma_2'$ finite such that

$$\frac{\Gamma_1'}{F_1} \cong \frac{\Gamma_2'}{F_2}$$

Intuition

Suppose **p** is some group theoretic property and Γ is a group. We say that Γ satisfies **p** **virtually** if Γ has a finite index subgroup satisfying **p**.

For example being *virtually abelian*.

Definition. **Quasi-rigidity**

A group Γ is called **quasi-rigid** if $\Gamma \cong_{\text{quasi}} \Gamma' \implies$ is virtually isomorphic to Γ'

Intuition

A group theoretic property **p** is called **quasi-rigid** if $\Gamma_1 \cong_{\text{quasi}} \Gamma_2$ and Γ_1 has property **p** implies there exists a Γ_3 with property **p** that is virtually isomorphic to Γ_2.

(Gromov's polynomial growth theorem, Milnor, Wolf) If Γ is quasi-isometric to a nilpotent group then Γ is virtually nilpotent.

(Stallings, Dunwoody et al) If $\Gamma \simeq_{\text{quasi}} F_n, n \geq 2$ then Γ is virtually free.

Example:

- $SL(2, \mathbb{Z}) = \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$ is virtually free.

Keywords

ends of groups, ends of metric spaces, chain of unbohnded components, throw balls, ...

(Gromov) Γ is quasi-isometric to $\mathbb{Z}^n$ implies Γ is virtually isomorphic to $\mathbb{Z}^n$.

$\Gamma \simeq_{\text{quasi}} \mathbb{Z}$ implies Γ is virtually cyclic

The first step is:

> **Proposition:** Γ is Gromov hyperbolic.

and then use:

Γ infinite Gromov hyperbolic $\implies$ Γ has an infinite order element.

Let $\gamma \in \Gamma$ be an infinite order element.

The second step is:

Let H hyperbolic group and $x \in H$ has infinite order. Then $\langle x \rangle \hookrightarrow H$ is a quasi-isometric embedding.

Hence $\mathbb{Z} \cong \langle \gamma \rangle \hookrightarrow_{\text{q-emb}} \Gamma \to_{\text{qi}} \mathbb{Z}$

- $\implies$ the composition is a qi

This implies

Example

$\langle \gamma \rangle \hookrightarrow \Gamma$ is a qi

This implies

> **Proposition:** $\Gamma / \langle \gamma \rangle$ is finite

proved by the fact that image of a quasi-isometry is coarsely surjective:

Exercise

If $f : X \to Y$ is a quasi-isometry then there exists $D \geq 0$ such that for all $y \in X$ $\exists x \in X$ such that

$$d(f(x), y) \leq D$$

(Dyabina?) Solvability is not a quasi-rigid property.

Examples of quasi-rigid properties:

- finite presentation
- Amenability
- hyperbolicity
- virtual cohomological dimension

Examples of quasi-rigid groups (quasi isometric implies virtually isomorphic):

- MCG of $S_g, g \geq 2$
- $\pi_1(S_g), g \geq 2$ (Gabai, Tukia, Cassen et al)
- π_1 of closed hyperbolic manifolds of dim ≥ 3 are quasi-rigid (Tukia, Sullivan)
- (Richard Schwartz) π_1 of finite volume hyperbolic ≥ 3-manifolds are quasi-rigid
- Suppose X is an irreducible symmetric space of non-compact type of rank ≥ 2
  - **(Klienier, Leeb) Uniform lattices in $\mathrm{Isom}(X)$ are quasi-rigid.**
  - **(Eskin) Non-uniform lattices of $\mathrm{Isom}(X)$ are quasi-rigid.**

A non-example:

- (Burger-Mozhes) $F_2 \times F_2$ is not quasi-rigid

Problems

- Are polycyclic groups quasi-rigid.
- Are $SL(2,\mathbb{Z}) \times SL(n,\mathbb{Z}), n \geq 3$ quasi-rigid
- Are hyperbolic groups the form F_n

Keywords

max cohomological dim of MCG

Margulis' superrigidity Theorem

Let $\Gamma \leq G = \mathrm{Isom}(X)$ acts on Y by isometry

$$\Gamma \to \mathrm{Isom}(Y)$$

then this action extends to a action of G on Y.

Question

Find similar theorem replacing Y by some other nice space

(Monad, Caprace, et al)

Generally this is false. There might be no nice action of Γ on Y.

Serre's property FA

Definition. **Serre's property FA**

A group Γ is said to have property if for any isometric action of Γ on a tree T there is a **global fixed point**.

(Serre) Suppose Γ is a countable group then Γ has FA $\iff$ Γ is fg, there is no surjective homomorphism $\Gamma \to \mathbb{Z}$ and Γ is not of the form $\Gamma_1 * \Gamma_2$.

We know $SL(2, \mathbb{Z})$ is an amalgam, but:

(Serre) $SL(3, \mathbb{Z})$ has FA.

proof Let $\Gamma = \langle \gamma_1, \ldots, \gamma_n \rangle$ act on some tree.

Step 1: $\mathrm{fix}(\gamma_i) \neq \emptyset$

Proposition: Given a nilpotent group $H \curvearrowright T$ acting on some tree

1. either H has a global fixed point, or there exists a group homomorphism $H \to \mathbb{Z} \leq \mathrm{Isom}(T)$ where $\mathbb{Z}$ acts by translation on the tree.
2. $[H, H] \curvearrowright T$ is trivial on the line

Step 2: $\mathrm{fix}(\gamma_i) \cap \mathrm{fix}(\gamma_j) \neq \emptyset$

Proposition: $\gamma_1 := I + E_{13}, \gamma_2, \gamma_3$ and

$$SL(3, \mathbb{Z}) = \langle \gamma_1, \gamma_2, \gamma_3 \rangle$$

Step 3:

Proposition: $H = \langle h_1, \ldots, h_n \rangle \curvearrowright T$ and $\mathrm{fix}(h_i) \neq \emptyset, \mathrm{fix}(h_i) \cap \mathrm{fix}(h_j) \neq \emptyset$ implies $\mathrm{fix}(H) \neq \emptyset$

Exercise

Step 2: $\mathrm{fix}(\gamma_i) \cap \mathrm{fix}(\gamma_j) \neq \emptyset$

$\implies \bigcap_i \mathrm{fix}(\gamma_i) \neq \emptyset$

Proposition: Any finite group has FA.

Culler-Votgmann's criterion

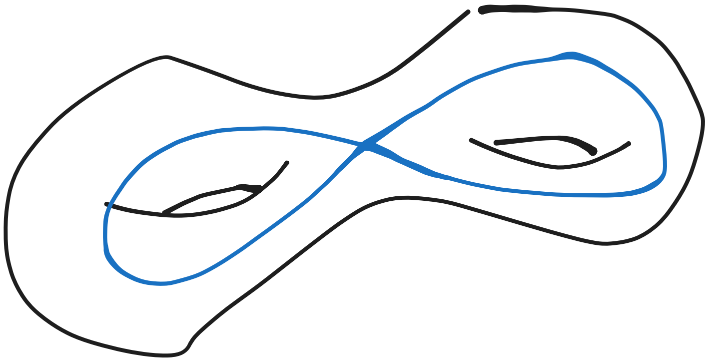
Let $\Gamma = \langle \gamma_1, \ldots, \gamma_n \rangle$ and $S = \{\gamma_1, \ldots, \gamma_n\}$. We construct a graph $\Delta(S)$ with vertex set S and edges

$$(\gamma_i, \gamma_j) \iff \exists \gamma_i^{\pm} \gamma_j^{\pm} \ldots \text{ or } \gamma_j^{\pm} \ldots \gamma_i^{\pm}$$

which commutes either with γ_i or γ_j. In particular if γ_i, γ_j commute.

Suppose $\Gamma \curvearrowright T$ $\Gamma = \langle S \rangle, \mathrm{fix}(\gamma_i) \neq \emptyset$ for all $\gamma_i \in S$ and $\Delta(S)$ is complete.

- *corollary:*
  - $SL(3,\mathbb{Z})$ has FA
  - MCG for $g \geq 2$ has FA
    - "non intersecting Dehn twists commute, intersecting curves braid relation satisfies our hypothesis"



  - $\mathrm{Aut}(F_n), n \geq 3$ has FA

property T

Definition. property T

We say that a group G has **property T** if for any action of

$$G \to U(\mathcal{H})$$

for a Hilbert space $\mathcal{H}$ with an almost invariant vector

$$\forall \epsilon > 0, K \subset_c G, \exists v \in \mathcal{H}, \| v \| = 1 : \|\gamma v - v\| < \epsilon, \forall \gamma \in K$$

then it has a non-zero invariant vector.

Examples

- compact groups

Non-examples

- $\mathbb{Z}$
- F_n

(Alperin) Any locally compact group Γ has property T $\implies$ Γ has property FA

Suppose Γ is a lattice in a semi-simple Lie group (of rank ≥ 2) with no rank 1 factor. Then Γ has property T.

(Kostant) $Sp(n,1)$ and any lattice in it has property T.

$SL(2,\mathbb{Z}) < SL(2,\mathbb{R})$ is a lattice in rank 1 semi-simple group which does not have propery FA.

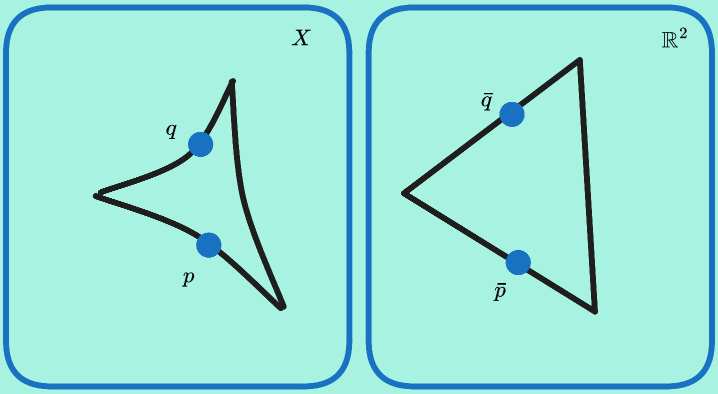
(V-Gelfand-Graev) Lattices in $SO(n,1), SU(n,1)$ do not have property T.

CAT(0) spaces

Consider a geodesic metric space X that is every pair of points have a length minimizing curve between them

$$\forall x, y \in X, \exists \alpha : [0, a] \to X, a(0) = x, \alpha(1) \\ d(\alpha(s), \alpha(t)) = |s - t|$$

Definition. **Consider a triangle in a metric space X and the corresponding triangle in $\mathbb{R}^2$ with standard metric with same side lengths. For $p, q \in X, \bar{p}, \bar{q} \in \mathbb{R}^2$ on the triangles the CAT(0)-inequality is**



$$d(p, q) \leq d(\bar{p}, \bar{q})$$

Definition. **A geodesic metric space that satisfies the CAT(0) inequality is called CAT(0) space.**

X is a complete simply connected Riemannian manifold with sectional curvature $\leq 0 \implies X$ is CAT(0)

Examples:

- symmetric spaces of non-compact type
- CAT(0) cube complexes
- Euclidean buldings
- Salveth complex "RAAG"

Definition. **(Farb) FA$_n$**

We say Γ has FA$_n$ if for any isometric action of Γ on a CAT(0) complex X of dim n Γ has a global fixed point.

Thus FA is same as FA$_n$.

Proposition: FA$_n \implies$ FA$_{n-1}$

(Farb) $SL(n, \mathbb{Z})$ for $n \geq 3$ has FA$_{n-2}$ (and this is the best).

(Helly) Suppose $\{X_i\}$ is a finite collection of open convex sets in $\mathbb{R}^n$ such that

$$X_{i_1} \cap X_{i_2} \cap \cdots \cap X_{i_{n-1}} \neq \emptyset$$

$\implies \bigcap_i X_i \neq \emptyset$

(Brindson) Suppose $\mathrm{MCG}(S_g), g \geq 2$ is acting by isometries on complete CAT(0)-space of dim $< g$. Then

$$\text{fix}(\text{MCG}) \neq \emptyset$$

(Bestma-Kapovich-Kleiner) Suppose X is a symmetric space of non-compact type and $\Gamma < \text{Isom}(X)$ is a lattice. Suppose Γ is acting properly discontinuously on a contractible n-manifold Y. Then $n \geq \dim(Y)$.

Pralay Chatterjee - Introduction to Lattices

Outline

- Generalities on lattices in Lie groups
- $SL(n, \mathbb{Z})$ is a lattice in $SL(n, \mathbb{R})$
- Borel density theorem
- state Borel-Harish-Chandra theorem
- Godment criteria for co-compactness of arithmetic lattices
- state Margulis Arithmeticity theorem for lattices in higher rank Lie groups
- Methods to produce arithmetic lattices

All groups will be second countable and locally compact. But mostly they will be Lie groups.

Every locally compact group G posses a left Haar measure and it is unique up to a multiplicative constant.

Definition. Modular function of a locally compact group

Let ν be a left invariant Haar measure on a locally compact group G Then we have the **modular function**

$$\Delta_G : G \to \mathbb{R}_{>0}$$
$$\Delta_G(g) := \frac{\nu(Ag^{-1})}{\nu(A)}$$

for any $A \in \mathfrak{B}_G$.

If G is a Lie group then we can choose a top form $\omega \in \Omega^{\dim G}(G)$. This gives a left invariant Haar measure.

Then it is easy to calculate the modular function:

Let $\mathfrak{g}$ be the Lie algebra of G. Then we have the adjoint representation

$$\text{Ad} : G \to GL(\mathfrak{g})$$
$$\mathfrak{g} \mapsto \text{d}_i c_g$$

and

$$\Delta(g) = |\det \text{Ad}(g)|$$

Definition. If $\Delta(G) = \{1\}$ then G is called unimodular.

Following groups are unimodular:

- compact groups
- G such that $G = [G, G]$
- semi-simple G
- connected nilpotent Lie group

However,

- solvable groups are **not** in general unimodular

All our measures are *always* regular.

Definition. ρ-semi invariant measures

Let

- $\rho : G \to \mathbb{R}_{>0}$ be a continuous character
- $(X, \mathfrak{M}, \mu)$ be a measure space
- $G \curvearrowright X$ be measurable group action

then μ is said to be **ρ-semi invariant** if $\mu(gA) = \rho(g)\mu(A)$ for all $g \in G, A \in \mathfrak{M}$.

Let $H < G$ be a closed subgroup of a Lie group G.

Then G/H is a manifold and

$$G \curvearrowright G/H$$

Let $H < G$ be a closed subgroup of a Lie group(?or locally compact?) G. Let $\rho : G \to \mathbb{R}_{>0}$ be a continuous character. Then $\exists$ ρ-semi invariant measure on G/H

$$\Longleftrightarrow$$

$\rho(h) = \Delta_G(h)\Delta_H(h^{-1})$ for all $h \in H$. Moreover, (given fixed ρ) if one of the above holds, any two such measures will be positive multiples of each other.

Reference

#book M S Raghunathan - Discrete Subgroups of Lie Groups - Springer (1972).pdf **#course/mini** Discrete subgroups of Lie groups - YouTube

- *(corollary)* If H is unimodular G/H admits a Δ-semi invariant measure
- *(corollary)* If $\Gamma < G$ is discrete, then G/H admits a Δ_G-semi invariant measure

Proposition: Suppose G is a Lie group and $H < G$ and H is unimodular. Suppose G/H has a G-invariant measure such that $\mu(G/H) < \infty$ then G is unimodular.

$$\mu\left(g\frac{G}{H}\right) = \Delta(g)\mu\left(\frac{G}{H}\right)$$

thus $\Delta(g) = 1$

$\Delta_G(H) = 1 \implies H \subset N := \ker \Delta_G$ and then we have

$$\pi : \frac{G}{H} \to \frac{G}{N}$$

- $\pi^*(\mu)$ is a measure on G/N
- $\implies \pi^*(\mu)$ is a finite Haar measure on G/N
- $\implies G/\ker \Delta_G$ is compact
- thus $\Delta_G(G) = 1$

Definition. **Let G be a Lie group and $\Gamma < G$ is a discrete subgroup then Γ is a lattice if G/Γ admits a G-invariant measure such that $\mu(G/\Gamma) < \infty$.**

Definition. **Let $\Gamma \subset G$ is discrete. Then a Borel set $F \subset G$ is called a fundamental domain for Γ if $G = F\Gamma$ and $F \cap F\gamma = \emptyset$ if $\gamma \neq i_G$.**

Let $\Gamma < G$ is any discrete subgroup then it admits a fundamental domain .

Let $U \subset G$ be open such that $U^{-1}U \cap \Gamma = \{i_G\}$

- G is second countable $\implies$

$$Ug_nU = G$$

- $$F := \bigcup_{n=1}^{\infty} \left(g_nU - \bigcup_{i<n} g_iU\Gamma \right)$$

Proposition: If F is a fundamental domain, $\gamma_i \in \Gamma$ and $F = \bigcup_{i\geq 1} A_i$ disjoint. Then

$$\bigcup_{i\geq 1} A_i\gamma_i$$

is another fundamental domain. Any fundamental domain can be obtained from another by such a process.

Proposition: If $C \subset G$ such that

$$\frac{C}{G} = \frac{G}{\Gamma} \iff C\Gamma = G$$

then there exists a fundamental domain $F \subset C$.

Proposition: If $B \subset G$ is such that $B \to \frac{G}{\Gamma}$ is injective then there exists a fundamental domain $F \supset B$.

Definition. **Let ν be the right invariant Haar measure on G and F be a fundamental domain for Γ. Then**

$$\nu_{G/\Gamma}(A) := v_G(\pi^{-1}(A) \cap F)$$

for all Borel $A \subset G/\Gamma$ and $\pi : G \to G/\Gamma$.

Exercise

Check that $v_{G/\Gamma}$ is independent of the fundamental domain chosen.

Use

Proposition: If F is a fundamental domain, $\gamma_i \in \Gamma$ and $F = \bigcup_{i\geq 1} A_i$ disjoint. Then

$$\bigcup_{i\geq 1} A_i\gamma_i$$

is another fundamental domain. Any fundamental domain can be obtained from another by such a process.

This measure is

- $B \subset G$ then $\nu_G(B) \geq \nu_{G/\Gamma}(\pi(B))$ and equal when π is injective on B
- *regular:* for compact $K \subset G/\Gamma$ choose $C \subset G$ compact such that $\pi(C) = K$ then

$$\infty > \nu_G(C) \geq v_{G/\Gamma}(K)$$

- Δ-semi invariant

1.12. Theorem. *Let G be a locally compact group satisfying the second axiom of countability and $\Gamma \subset G$ a lattice. Let $\{x_n\}_{1\leqq n<\infty}$ be a sequence in G. Let $\pi: G \to G/\Gamma$ be the natural map. Then the sequence $\pi(x_n)$ has no convergent subsequence in G/Γ if and only if there exists a sequence $\{\gamma_n\}_{1\leqq n<\infty}$ in Γ such that $\gamma_n \neq e$ for any n and $x_n\gamma_n x_n^{-1}$ converges to e as n tends to ∞.*

$SL(n,\mathbb{Z})$ is a lattice in $SL(n,\mathbb{R})$

$SL(n,\mathbb{Z})$ is a lattice in $SL(n,\mathbb{R})$

proof: To get a Borel subset C such that

$$SL(n,\mathbb{R}) = C\Gamma$$

such that $\mu(C) < \infty$.

Proposition: $SL(n,\mathbb{R}) = KA_{\sqrt{2}/3}N_{1/2}\Gamma$

Let G be a Lie group and $S, T \leq G$ be closed, $T \cap S$ compact with $S \times T \to G$ product map open and surjective. Then

$$\int f\delta\mu_G = \int_{S\times T} f(st)\frac{\Delta_T(t)}{\Delta_G(t)}\delta\mu_S(s)\delta\mu_H(t)$$

Riddhi Shah - The structure of automorphism groups and lattices in Lie groups

Abstract

We study the structure of groups of automorphisms of a connected Lie group; namely, we identify certain conditions under which they are almost algebraic, and discuss some applications and examples (joint work with S.G. Dani - https://arxiv.org/abs/2504.18641). We explore the structure of lattices in a connected Lie group and discuss some properties of automorphisms which keep a lattice invariant (joint work with Rajdip Palit and Manoj B. Prajapati - Geometry, and Dynamics 17 (2023), 185-213 - https://doi.org/10.4171/GGD/672)

$$\mathrm{Aut}(G) \to \mathrm{AutLieAlg}(\mathfrak{g}) \subset GL(\mathfrak{g})$$

is one to one

$$\exp(\mathrm{d}\tau(X)) = \tau(\exp X)$$

Definition. $\mathrm{Aut}(G)$ **is almost algebraic if it is open subgroup finite index in an algebraic group.**

(Wigner, 1976) $\mathrm{Aut}(G)_0$ **is almost algebraic.**

(Dani 1992) Let C be the max cpt connected central (max central tori) subgroup of G**. Then** $\mathrm{Aut}(G)$ **is almost algebraic if** C **is trivial.**

	Aut
T^n	$GL(n, \mathbb{Z})$
$\mathbb{R}^n$	$GL(n, \mathbb{R})$?
simply connected G	AutLieAlg($\mathfrak{g}$)? Zariski closed? algebraic...
G/C as in **(Dani 1992) Let C be the max cpt connected central (max central tori) subgroup of** G**. Then** $\mathrm{Aut}(G)$ **is almost algebraic if** C **is trivial.**	$\mathrm{Aut}(G/C)$ is almost algebraic
N_3	
$SL(2, \mathbb{R})$\semi$G$	almost alg
$SO(2, \mathbb{R})$\semi G	not alg!

....

C S Rajan - Harmonic super-rigidity of Corlette and Gromov-Schoen

III. Vector bundle valued harmonic forms

(1) Betti numbers of locally symmetric spaces, [24, 25, 28, 31].

In 1960, Matsushima returned to his Alma Mater, Osaka University, to fill the Chair of Algebra vacated by his teacher, Professor Shoda. It was in Osaka that he wrote a series of important papers on cohomology of locally symmetric spaces.

According to Kodaira's theory of deformations, the space of infinitesimal deformations of a pseudogroup structure is given in terms of the first cohomology with coefficients in the sheaf of germs of infinitesimal automorphisms. On the other hand, this cohomology group can be represented by harmonic forms by the Hodge theory. Using Bochner's technique of proving vanishing theorems, Calabi and Vesentini [1] obtained vanishing of certain cohomology groups $H^q(M, \Theta)$ for compact quotients M of symmetric bounded domains and their sheaves Θ of germs of holomorphic vector fields. Using the same idea, Weil

[1] proved the infinitesimal rigidity for uniform discrete subgroups of semisimple Lie groups.

In [24], Matsushima proved that if M is a compact quotient of a symmetric bounded domain X none of whose irreducible factors is equivalent to the unit ball in $\boldsymbol{C}^m$, then the first cohomology $H^1(M, \boldsymbol{R})$ vanishes. This brought out in striking contrast the difference between the classical 1-dimensional case and the higher dimensional case. As in the work of Calabi-Vesentini and Weil, positivity of a certain quadratic form defined by the curvature is the essential point of the proof.

In [25], Matsushima considered higher dimensional cohomology $H^q(M;\boldsymbol{R})$. If X^* is a symmetric space of compact type, then $H^q(X^*;\boldsymbol{R})$ or the space of harmonic q-forms on X^* is given, according to E. Cartan, by invariant parallel q-forms on X^*. These q-forms induce invariant parallel q-forms on the noncompact dual X which, in turn, induce parallel (and hence, harmonic) q-forms on the quotient M. Hence, $\dim H^q(M;\boldsymbol{R}) \geqq \dim H^q(X^*;\boldsymbol{R})$ in general. Considering again positivity of a similar quadratic form, Matsushima determined the range of q for which the equality $H^q(M;\boldsymbol{R}) = H^q(X^*;\boldsymbol{R})$ holds for each symmetric bounded domain X. Kaneyuki and Nagano [1], [2] completed Matsushima's program by treating the remaining symmetric spaces X of noncompact type. Matsushima's lecture at the Bombay Colloquium [28] is a survey of his and their results on Betti numbers of locally symmetric spaces.

Matsushima's paper [31] which relates Betti numbers of locally symmetric spaces to unitary representations is of growing importance. Let $X = G/K$ be a symmetric space of noncompact type and $M = \Gamma\backslash X$ a compact locally symmetric space. In [31] he gave a formula for the Betti numbers of M in terms of the multiplicities of certain unitary representations of G in the right regular representation of G on $L^2(\Gamma\backslash G)$. Hotta and Wallach [1] investiagte existence and non-existence of unitary representations appearing in the Matsushima formula and explain vanishing theorems in Matsushima's earlier papers [24], [25].

We consider families of complex manifold:

$$\begin{array}{ll} \mathcal{X} & \text{total space} \\ \big\downarrow{\scriptstyle \pi} & \\ S & \text{parameterizing space} \end{array}$$

the fibres $\mathcal{X}_s := \pi^{-1}(s)$ is space with *extra structure*.

Intuition

A space $X_0 \cong \mathcal{X}_{s_0}$ is **locally rigid** if for every s in a sufficiently small nbd of $s_0 \in S$

$$\mathcal{X}_s \cong \mathcal{X}_{s_0} \cong X_0$$

This is *hard* to verify. Instead, we replace this with infinitesimal or cohomological condition in terms of X_0.

Ehresmann's theorem for smooth manifolds

(Ehresmann) Let $\mathcal{X}, S$ are manifolds and

$$\begin{array}{c} \mathcal{X} \\ \big\downarrow \pi \\ S \end{array}$$

is be a proper, smooth submersion. Then locally on S, π is a product

$$\begin{array}{ccc} \pi^{-1}(U) = \mathcal{X}_U & \to & X_{s_0} \times U \\ \big\downarrow & & \big\downarrow \\ U & = & U \end{array}$$

proof by implicit function theorem we have

- the vector field

$$\frac{\partial}{\partial t} \text{ on } U$$

  and pull back by π_* and patch using partition of unity
- ...

[1]

for complex manifolds

The partition of unity argument fails.

[2]

1-cocycle of holomorphic vector fields on the fibre X_0

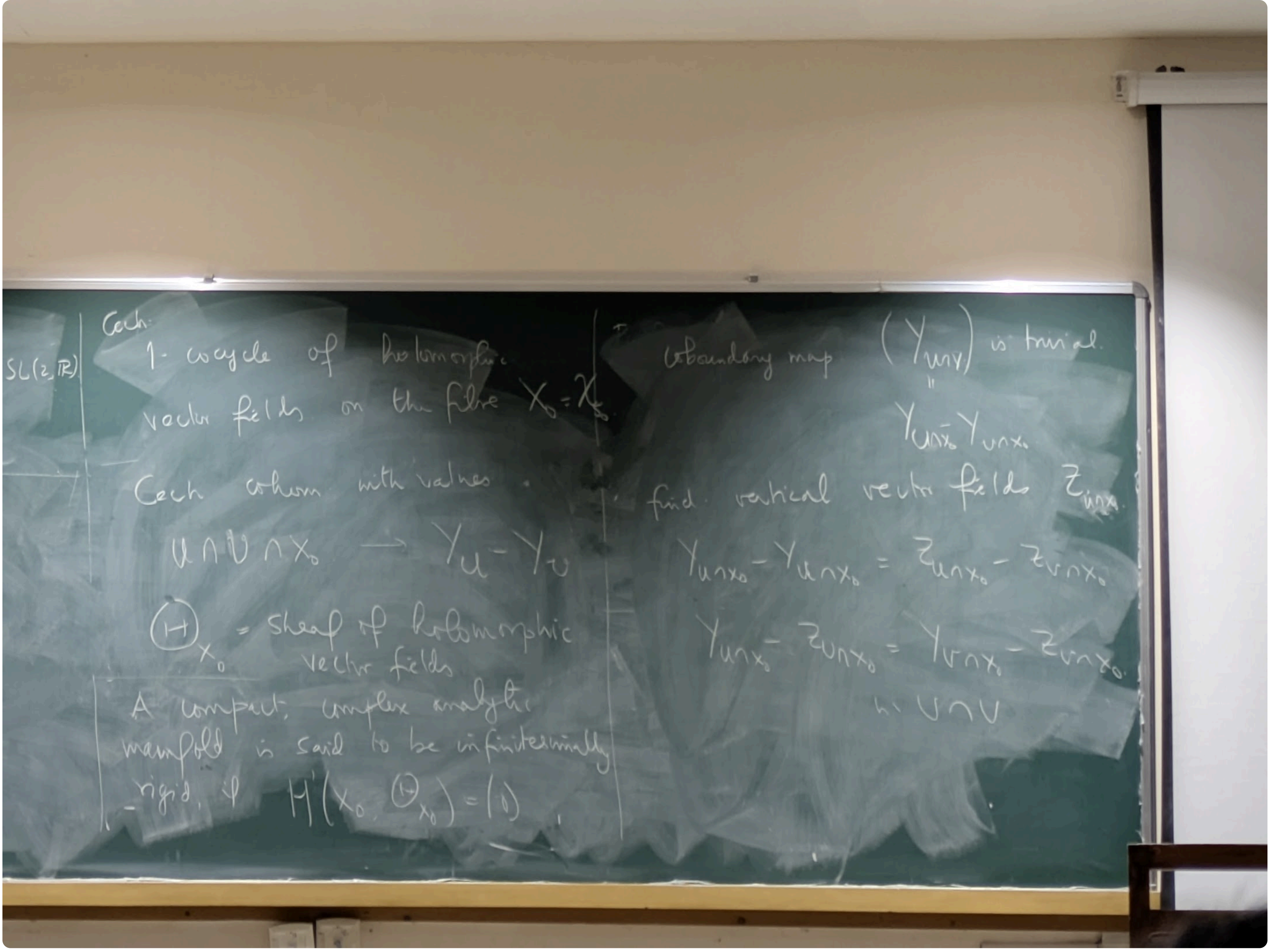
Cech cohomology with values

...

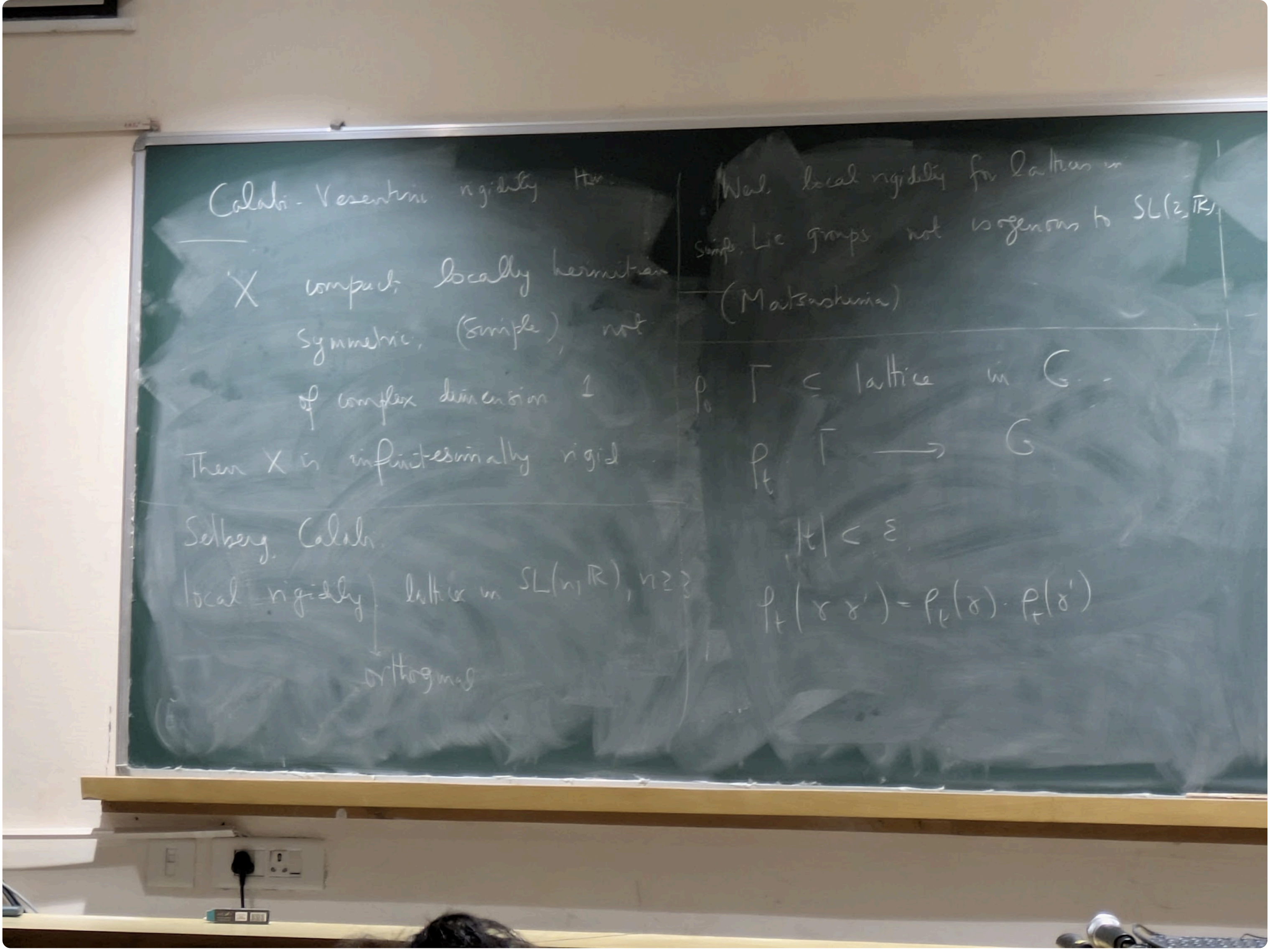
Definition. **Infinitesimal rigidity**

A compact, complex manifold is said to be **infinitesimally rigid** if

$$H^1(X_0, \mathscr{O}_{X_0}) = (0)$$



Calabi-Visentini rigidity theorem



Let X be compact, locally Hermitian symmetric spaces of dim > 1 then X is infinitesimally rigid.

local rigidity:

there exists $g_t \in G$, such that

$$\rho_t = g_t \rho_0 g_t^{-1}$$

1-cocycle valued in $\mathfrak{g} = \mathrm{Lie}(G)$ of Γ

Tangent space at ρ_0

$$Z^1_{\rho_0}(\Gamma, \mathfrak{g}) \to \text{1-cocycles}$$

Γ is finitely presented

S set of generators for Γ

G^S $\qquad G^S = \mathrm{Maps}(S, G)$

$s \in S$ $\quad \rho_t(s) \in G$

$\rho_t(S) \subseteq G^{|S|}$ $\qquad s_1 s_2 = s_2 s_1$

finitely many relations

$\mathrm{Rep}(\Gamma, G) \subseteq G^S$

cut out by the relations

$\Gamma \to \mathfrak{g}$

$$c(\gamma) = \frac{d}{dt}\Big|_{t=0} \left(\rho_t(\gamma)\rho_0(\gamma)^{-1}\right)$$

$$\rho_t(\gamma\gamma') = \rho_t(\gamma)\cdot\rho_t(\gamma')$$

$c(\gamma\gamma') = ?$

$$\rho_t(\gamma)\left(\rho_t(\gamma')\rho_0(\gamma')^{-1}\right)\rho_0(\gamma)^{-1}$$

$$= \rho_t(\gamma)\left(\rho_t(\gamma')\rho_0(\gamma')^{-1}\right)\rho_t(\gamma)^{-1}\rho_t(\gamma)\rho_0(\gamma)^{-1}$$

$t=0$: $$c(\gamma\gamma') = c(\gamma) + \rho_0(\gamma)c(\gamma')\rho_0(\gamma)^{-1}$$

$$\in Z^1_{\rho_0}(\Gamma, \mathfrak{g})$$

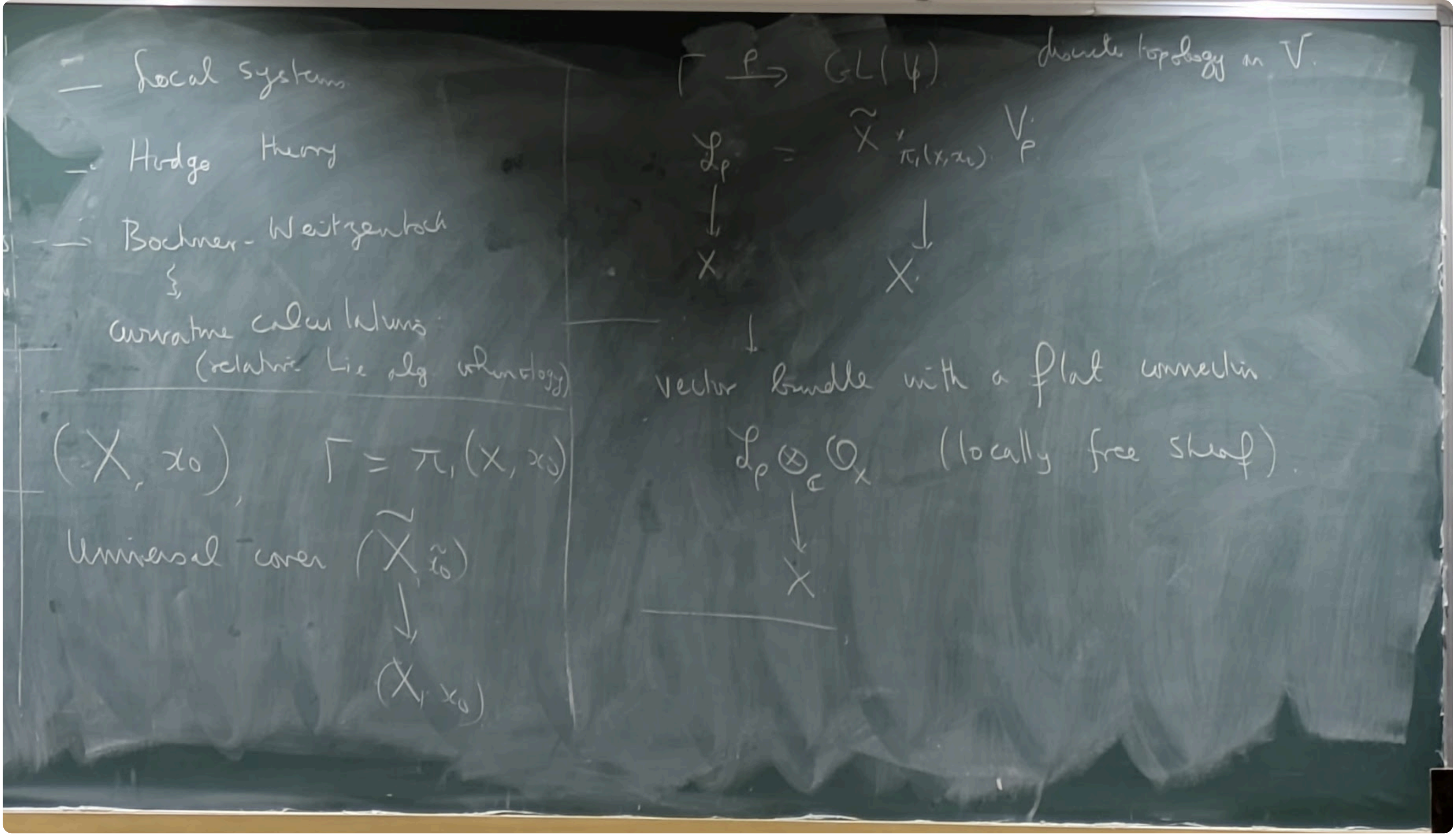
1-coboundary $\exists\, a \in \mathfrak{g}$

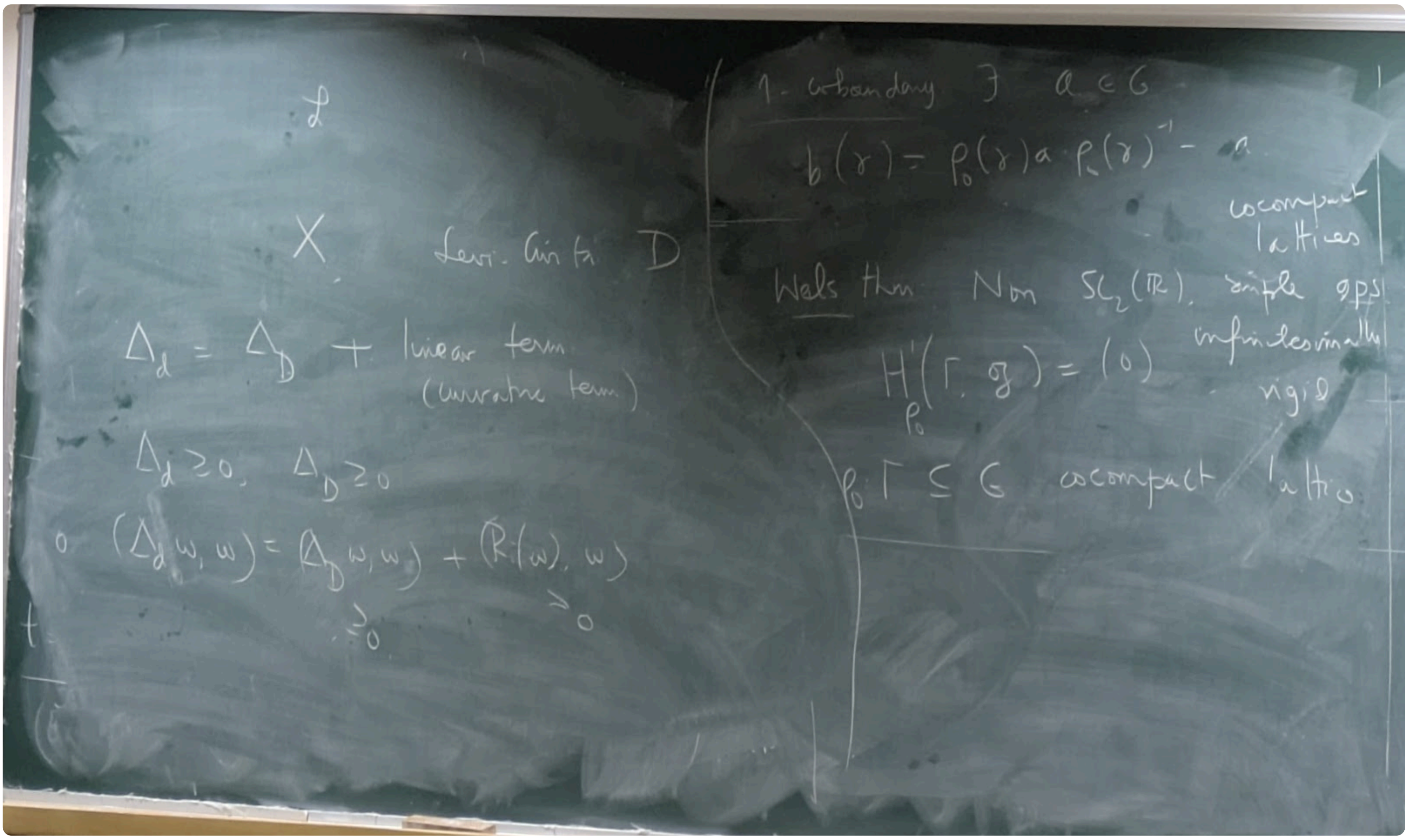
$$b(\gamma) = \rho_0(\gamma)\, a\, \rho_0(\gamma)^{-1} - a$$

Weil's thm: Non $SL_2(\mathbb{R})$, simple gps, cocompact lattices infinitesimally rigid

$$H^1_{\rho_0}(\Gamma, \mathfrak{g}) = (0)$$

$\rho_0 : \Gamma \subseteq G$ cocompact lattice





$$\{\rho : \Gamma \to GL(V_\rho)\}$$

? $\leftrightarrow$ local systems (discrete topology on V_ρ) $\leftrightarrow$ vector bundles with flat connection (standard topology on V_ρ)

Bochner's method

Let M be a Riemannian manifold and

$$\begin{array}{c}(E, \langle,\rangle, D)\\ \big\downarrow \\ (M, g)\end{array}$$

$$X \in \mathrm{Vec}(M)$$

- D is a connection, so $s \in \boldsymbol{\Gamma}(U, E)$ then

$$D_X s \in \boldsymbol{\Gamma}(U_1, E)$$

- D is *compatible with M* that is

$$X\langle s, t\rangle = \langle D_X s, t\rangle + \langle s, D_X t\rangle$$

- E also has other natural operators
  - if E is associated to a local system $\mathscr{L}_\rho \to M$ then

$$\begin{array}{c}E_\rho = \mathscr{O}_M \otimes \mathscr{L}_\rho\\ \big\downarrow \\ M\end{array}$$

    then it has a connection

$$D_{\mathscr{L}} : E_\rho \to T^*M \otimes E_\rho$$

  - (Weitgenboch) E is the vector bundle of p-forms $p \geq 1$

$$\boldsymbol{\Lambda}^p T^*M \xrightarrow{\mathrm{d}^p} \boldsymbol{\Lambda}^{p+1} T^*M$$

  - *Laplacian*

$$E \xrightarrow{D} \Omega^1 \otimes E \xrightarrow{D} \Omega^2 \otimes E \to \dots$$

$$\Delta_D = D^*D + DD^*$$

  - *Bochner-Laplacian*

$$\Delta := DD^*$$

Lie algebra cohomology

$$\mathfrak{g} \to \mathrm{End}(M)$$

we define

$$H^*(\mathfrak{g}, M)$$

which has a usual co-chain model

$$\{f : \Lambda^i(\mathfrak{g}) \to M\} =: C^i(\mathfrak{g}, M) \xrightarrow{d} C^{i+1}(\mathfrak{g}, M)$$

where

$$\mathrm{d}f(x_1, \dots, x_{i+1}) = \sum (-1)^{i+1} x_i f(x_1, \dots, \hat{x}_i, \dots, x) + \sum$$

where

$$H^i = \frac{\ker}{\operatorname{im}}$$

Relative Lie algebra cohom

$K \leq G, \mathfrak{k} \leq \mathfrak{g}$

Bernstein

$$R(\mathfrak{g}, K) := \text{distributions on } G \text{ supp in } K$$

where K is compact and the algebra is under convolution, this is the *Hecke algebra at infinity*

Out $(\mathfrak{g}, K)$-modules $\leftrightarrow$ $R(\mathfrak{g}, K)$-modules

$$C^i(\mathfrak{g}, K, M) := K\text{-equivariant maps } \Lambda^i(\mathfrak{g}/\mathfrak{k}, M)$$

$$f(kx_1, \ldots, kx_i) = \rho(k)^{-1} f(x_1, \ldots, x_i)$$

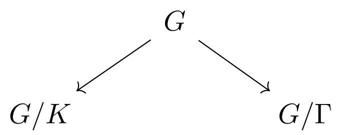
---

semi-simple $G \geq K$ maximal compact with

$$\rho : G \to \operatorname{Aut}(V_\rho)$$

$M = V_\rho$ finite dim then

$$H^*(\Gamma, \rho) \cong H^*\left(\mathfrak{g}, K, \mathcal{C}^\infty\left(\frac{G}{\Gamma}\right) \otimes V_\rho\right)$$



$$(\Delta^p \eta)(x_i)$$

The chronology of rigidity begins with the theorem of A. Selberg ([16]) that a discrete co-compact subgroup Γ of $SL(n, \mathbf{R})$ cannot be continuously deformed except trivially, that is, by inner automorphisms of $SL(n, \mathbf{R})$, if $n > 2$; Selberg's proof rested on showing that the trace of elements in Γ are preserved under deformations of Γ. Selberg's method applied to the other classical groups of rank greater than 1. At about the same time, E. Calabi and E. Vesentini proved the rigidity of complex structure under infinitesimal deformations of compact quotients of bounded symmetric domains ([3b]), and later Calabi proved the metric analogue for compact hyperbolic n-space forms for $n > 2$ ([3a]). Thereupon A. Weil ([21]) generalized Selberg's and Calabi's results to semi-simple groups having no compact or 3 dimensional simple factors. Weil's proof deduces the rigidity of Γ in G under deformations from the vanishing of the cohomology group $H^1(\Gamma, \dot{G})$ where $\dot{G}$ is the Lie algebra of $\dot{G}$ regarded as a Γ-module under the adjoint representation. In the case of arithmetic subgroups, which are lattices (that is, Γ is discrete in G and G/Γ has finite Haar measure) but not generally co-compact, the rigidity under deformations was proved independently by A. Borel (unpublished) in the case of $\mathbf{Q}$-simple groups of $\mathbf{Q}$-rank at least two, and H. Garland (in the split case), and by M. S. Raghunathan in the remaining $\mathbf{Q}$-rank one cases, again by showing that $H^1(\Gamma, \dot{G}) = 0$.

Margulis super-rigidity

Following: Venkataramana-rigidity-notes.pdf

Tldr

super-rigidity $\implies$ Arithmeticity

Let $G = PSL(n, \mathbb{R}), n \geq 3$.

We want to show Γ is arithmetic from this theorem:

(superrigidity) Let $L = \mathbb{R}, \mathbb{C}, \underbrace{K \geq}_{\text{finite}} \mathbb{Q}_p$ (locally compact fields of char 0) and H be a algebraic group of adjoint type. Let G is non-compact simple Lie group of adjoint type, Γ be a fg lattice (so irred) in G. A rep

$$\rho : \Gamma \to H(L)$$

continuous with $\rho(\Gamma)$ is non-compact. Then ρ extends to a continuous map

$$G \to H(L)$$

Marc Bourdon - Lie groups and quasi-isometries

Abstract

In geometric group theory, one studies groups as geometric objects, and tries to determine whether an algebraic property is a geometric one. In this setting, the natural maps between groups are the so-called quasi-isometries (QI). A class of groups is said to be QI-rigid if every finitely generated group that is QI to a group in the class, is virtually isomorphic to (another) group in the class.

Geometric group theory started in the 80's when Gromov proved that the class of nilpotent groups is QI-rigid. In the 90's the subject focused on semisimple groups; it culminated when a combination of works allowed to establish that the class of lattices in a given semisimple group is QI-rigid.

Since then, geometric group theory has developed several interactions with other fields of mathematics. It has been used to solve some old open problems (e.g. in 3-manifold topology).

A major problem in geometric group theory is to classify connected Lie groups up to QI. One knows that every simply connected Lie group is QI to a completely solvable Lie group, i.e. to a closed subgroup of the upper triangular real matrix group. In 2018 Cornulier conjectured that two QI completely solvable groups must be isomorphic. This is currently open, even in the smaller class of nilpotent Lie groups.

During the talks, I plan to introduce some QI-invariants for Lie groups, and illustrate them with examples. These invariants include the growth rate, the rank, and the L^p-cohomology. The examples of groups that will be discussed include the nilpotent groups, the Heintze groups, the abelian-by-abelian solvable Lie groups. Some known results about their QI classification will also be presented.

Let G be a finitely generated or connected Lie group.

We geometrize G:

- If G is fg let S be a finite set of generators

  $$|g|_S := \min\{k \mid g = s_1 \dots s_k, s_i \in S \cup S^{-1}\}$$

  and

  $$d_S(g, g') := \left|g^{-1} g'\right|$$

  is a metric on G
- If G is a Lie group, we equip G with a left invariant Riemannian metric

  $$\langle u, v\rangle_g := \left\langle L_{g^{-1}*} v, L_{g^{-1}*} w\right\rangle_{i_G}$$

Tldr

We want to study large scale geometry of G, that is, study upto quasi-isometry.
Thus we need quasi-isometry invariants!

(Gromov: *Asymptotic invariants of infinite groups*)

growth

Definition. Growth of fg groups

If G is finitely generated with S a finite set of generators, the **growth function** of (G, S) is

$$\begin{aligned}\rho_{G,S} : \mathbb{R} &\to \mathbb{R}_{>0} \\ r &\mapsto \left| \underbrace{\{g \in G \mid |g|_S \leq R\}}_{\overline{B(i,R)}} \right|\end{aligned}$$

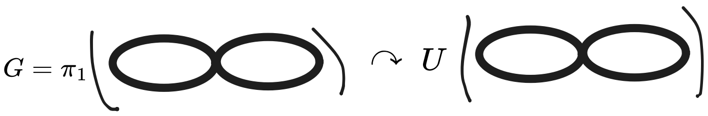
Definition. Growth of Lie groups

If G is a Lie group equipped with a left-invariant Riemannian distance and h be a left Haar measure on G then the **growth function** of (G, d, h) is

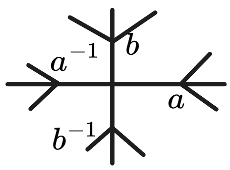
$$\begin{aligned}\rho_{G,d,h} : \mathbb{R}_{>0} &\to \mathbb{R}_{>0} \\ \rho(R) &= h(B_d(1, R))\end{aligned}$$

- $(\mathbb{R}^n, d_{\text{std}})$ has growth

$$d(r) = cr^n$$



4-valence tree

- 

$\text{Cal}(G, \{a, b\})$

    - $$\rho(n) = 4 \sum_{k=1}^{n} 3^{k-1} + 1$$

Definition. Equivalence of growth functions

Let $\rho_1, \rho_2 : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$

- we say $\rho_1 \preceq \rho_2$ if $\exists a, b > 0, c, d \geq 0$

$$\rho_1(r) \leq a\rho_2(br + c) + d$$

- and $\rho_1 \simeq \rho_2$ if $\rho_1 \preceq \rho_2$ and $\rho_2 \preceq \rho_1$

Exercise

$r^d \simeq r^D \implies d = D$

Exercise

$e^{\alpha r} \simeq e^{\beta r}$ for all $\alpha, \beta > 0$

Let G, H be fg or Lie groups. Then G QI to H implies $\rho_G \simeq \rho_H$.

Proposition: Let G be fg.

- if $H < G$ is a fg subgroup then $\rho_H \preceq \rho_G$
- if $N \trianglelefteq G$ then $\rho_{G/N} \preceq \rho_G$

Exercise

State an prove analogous result for Lie groups.

Proposition: Let G be a Lie group

- either
  - *polynomial growth* $\rho_G \preceq r^d$
  - *exponential growth* $\rho_G \simeq e^r$

Proposition: (Frank's lemma) Suppose G admit an automorphism $h : G \to G$ which is expanding, that is,

$$\exists \lambda > 1 : \forall g, g' \in G \; d(h(g), h(g')) > \lambda d(g, g')$$

then G has polynomial growth.

proof

$$B(i, \lambda^k) \subset h^k B(i, 1)$$

- Since h is an automorphism, $h^* \nu = c\nu$
- $\implies \rho(\lambda^k) = \nu(B(i, \lambda^k)) \leq h^{k*}\nu(B(i,1)) = c^k \nu(B(i,1))$
- $\implies \rho(r) \preceq r^d$

Example

$$G = \left\{ \begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \right\} \subset GL(n, \mathbb{R})$$

$$\begin{aligned} h_t : G &\to G \\ (a_{ij}) &\mapsto e^{t(j-i)}(a_{ij}) \end{aligned}$$

is an automorphism and for $t > 0$ its expanding.
Thus G has polynomial volume growth.

Proposition: If G is not unimodular, then G has exponential growth.

Example

$G = SL(n, \mathbb{R}) = KAN$
K is compact $\implies$ G is QI to AN which is not unimodular
Thus G has exponential growth.

(Wolf) Every nilpotent (fg or Lie) group has polynomial growth.

- *proof:* by Ado-Engel's theorem

(Malev) Every fg nilpotent group admits a finite index subgroup which is isomorphic to a cocompact lattice in a nilpotent Lie group.

(Gromov) Let G be a fg group. If G has polynomial growth, then it admits a finite index subgroup which is nilpotent.

Exercise

$\mathbb{R} \ltimes_{R^\theta} \mathbb{R}^2$ has polynomial growth (QI to $\mathbb{R}^3$) but is not nilpotent.

Proposition: A Lie group of polynomial growth is always QI to a nilpotent Lie group.

- *corollary of*

  (Gromov) Let G be a fg group. If G has polynomial growth, then it admits a finite index subgroup which is nilpotent.

  Let G, H be fg, G is QI to H and H is nilpotent. Then G is virtually nilpotent.

Nilpotent groups and solvable groups

Examples:

- $$N := \left\{ \begin{bmatrix} 1 & & * \\ & 1 & \\ & & 1 \end{bmatrix} \right\}$$

  is nilpotent

- $$T := \begin{bmatrix} a_{11} & & * \\ & \ddots & \\ & & a_{nn} \end{bmatrix}$$

  is solvable as $[T, T] = N$

(Gromov) Let G be a fg group. If G has polynomial growth, then it admits a finite index subgroup which is nilpotent.

and

(Malev) Every fg nilpotent group admits a finite index subgroup which is isomorphic to a cocompact lattice in a nilpotent Lie group.

implies

(Ado-Engel-Malev) If a group is fg nilpotent, then it is virtually linear.

are false for fg solvable groups by following:

(Erscher) G, H fg, G quasi-isometric to H but H solvable does not imply G is virtually solvable.

Consider the lamplighter group

$$G := \mathbb{Z} \ltimes_\varphi \bigoplus_{\mathbb{Z}} \mathbb{Z}_2$$

where

$$\begin{aligned}\varphi : \mathbb{Z} &\to \mathrm{AutGrp}\left(\bigoplus_{\mathbb{Z}} \mathbb{Z}_2\right)\\ (x_i) &\mapsto (x_{i+n})\end{aligned}$$

where the multiplication is

$$(n, x)(m, y) = (n + m, x + \varphi(n)y)$$

This group is generated by

$$(1, 0), (0, (\dots 0, 0, 0, \underbrace{1}_{0\text{-th}}, 0, 0, 0 \dots)$$

This group is solvable as

$$[G, G] = \bigoplus_{\mathbb{Z}} \mathbb{Z}$$

A fg linear group is virtually torsion free but this group isn't. Thus this grouo us not virtually linear.

Distortion

Definition. (Gromov) Distortion

Let G be a fg or Lie group. An element $g \in G$ is

- **distorted** if

$$\frac{d(i, g^n)}{n} \xrightarrow{n\to\infty} 0$$

- **exponentially distorted** if $\exists c > 0$ such that $\forall n \in \mathbb{N}$

$$d(i, g^n) \leq C \log(1 + n)$$

- **infinitely distorted** if $\exists c > 0$ such that $\forall n \in \mathbb{N}$

$$d(i, g^n) \le C$$

Examples:

- Consider $G = SL(n, \mathbb{R})$ and a norm $\| \ \|$ on $\mathrm{Mat}(n, \mathbb{R})$ then

$$l(g) := \log \max\{\|g\|, \|g^{-1}\|\}$$

  there exists $c \ge 1, D \ge 0$ such that $\forall g \in G$

$$\frac{1}{c} l(g) - D \le d(i, g) \le c l(g) + D$$

  - g semi-simple $\neq i \implies g$ is non-distorted
  - g unipotent $\implies g$ is exponentially distorted
  - $g \in SO(n, \mathbb{R}) \implies g$ is infinitely distorted

Suppose G is a Lie group. Then G admits a non-infinitely exponentially distorted element $\iff G$ is of exponential growth.

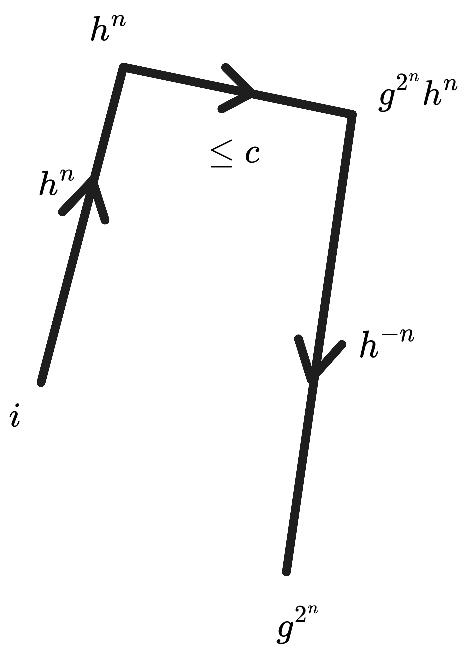
proof: G is of polynomial growth $\iff X \in \mathfrak{g}$ has $\mathrm{spec}(\mathrm{ad}(X)) \subset i\mathbb{R}$

- $\implies$ every $g \in G$ is at most polynomially distored
- if G is not of polynomial growth then $\exists\ H \in \mathfrak{g}$ such that $\mathrm{ad}H$ admits an eigenvalue λ with $\Re(\lambda) \neq 0$

Proposition: Let $X \in \mathfrak{g}$ be an eigenvector of $\mathrm{ad}H$ of eigenvalue $\lambda \in \mathbb{R} \setminus \{0\}$ then $g = \exp(X)$ is exponentially distorted.

Assume $\lambda = \log 2$ then $h = \exp H$.

- **Proposition:** $d(i, h^{-n} g^{2^n} h^n) \le c_0$ for $n \gg 1$

  - 

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Exponential radical

Definition. (Osin) Exponential radical

Let G be a Lie group. Its exponential radical denoted by $R_{\exp}G$ is the set of exponentially distorted elements of G.

(Guivarc'h, Osin) Suppose G is simple connected solvable Lie group. Then $R_{\exp}G$ is a closed normal subgroup of G.Moreover it is the smallest normal closed subgroup H of G such that G/H has polynomial growth.

Here, we cannot drop the assumption of *solvable*. For example $G = SL(2,\mathbb{R})$ and $g_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $g_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ are exponential distorted but $g_1 g_2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ is not since its semi-simple.

Arghya Mondal - Mostow rigidity for closed hyperbolic 3-manifolds

Symmetric spaces of non-compact type Y are of the form

$$Y = X/\Gamma, X = G/K$$

where G is a non-compact semi-simple Lie group, no compact factor and no centre, with $K < G$ compact, $\Gamma < G$ torsion free lattice.

In its more generality, we have the rigidity theorem:

(Mostow-Prasad rigidity) Let Y_1, Y_2 are locally symmetric spaces of non-compact type, with no $\mathbb{R}H^2$ factor with an isomorphism

$$\varphi : \pi_1(Y_1) \to \pi_1(Y_2)$$

isomorphism. Then there exists an (upto scaling) isometry

$$f : Y_1 \to Y_2, f_* = \varphi$$

Mostow proved this for

- closed hyperbolic manifolds
- general case with cocompact lattices

Then Prasad proved for

- $\mathbb{Q}$-rank 1 lattices (non-compact type in rank 1 symmetric spaces)

Let $G(\mathbb{Q})$ be an algebraic group over $\mathbb{Q}$. Then the maximal torus in $G(\mathbb{Q})$ is of dimension 1 is what "$\mathbb{Q}$-rank 1" means.

The version we will prove is for closed hyperbolic 3-manifolds.

Let Y be a *complete* hyperbolic manifold, that is, it has constant sectional curvature -1.

Uniformization theorem implies the universal cover $\tilde{Y}$ must be $\cong \mathbb{R}H^n$ and

$$\Gamma := \pi_1(Y, y) \curvearrowright \mathbb{R}H^n$$

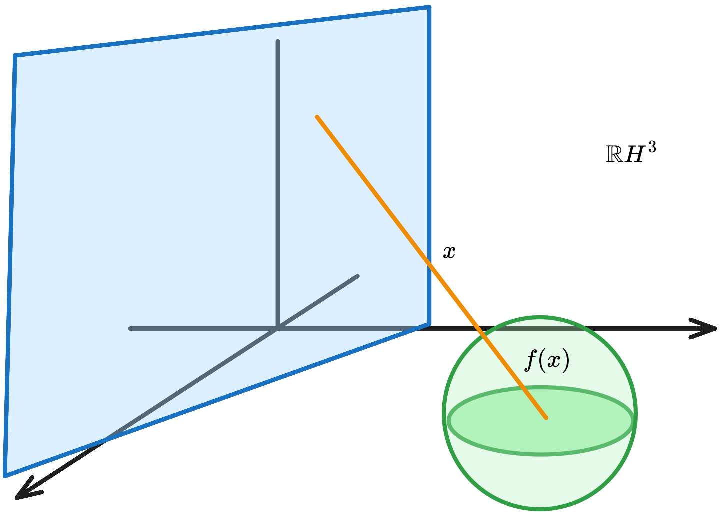
via Deck transformations this

$$Y = \frac{\mathbb{R}H^n}{\Gamma}$$

Isometries are

$$O^+(1,3) \curvearrowright \mathbb{R}H^n$$

which are generated by *reflections*



such that

$$\|f(x)\| \; \| x \| \; = R^2$$

The boundary

$$\partial \mathbb{R}H^3 := \mathbb{R}^2 \cup \{\infty\} \cong \mathbb{C}P^1$$

where Mobius transformations along with reflections act induced by the hyperbolic isometries

$$PSL(2,\mathbb{C}) \ltimes \mathbb{Z}_2 \curvearrowright \mathbb{C}P^1$$

and we have

$$\text{Isom}^+(\mathbb{R}H^3) \cong SO^+(1,3) \cong PSL(2,\mathbb{C})$$

We have projection map to a totally geodesic submanifold P

$$\pi_P : H^3 \to P$$

such that the geodesic $[x, \pi_P(x)]$ is perpendicular to P for all $x \in H^3$ and

$$d(\pi_P(x), \pi_P(y)) \leq d(x,y)$$

So in particular

- $$\Gamma < O^+(1,3)$$
-

(Mostow rigidity for closed 3-manifold) **Let M_1, M_2 be oriented closed hyperbolic 3-manifolds with $\Gamma_1 := \pi_1(M), \Gamma_2 := \pi_2(M)$ and a isomorphism**

$$\varphi : \Gamma_1 \to \Gamma_2$$

Then there exists an isometry $f : M_1 \to M_2$ with $f_* = \varphi$.

Tldr

1. Let

$$\varphi : \Gamma_1 \to \Gamma_2$$

be a group homomorphism. This will induce a quasi-conformal

$$\partial H^3 \to \partial H^3$$

equivariant with the action of Γ_i on the boundary.
2. Any *equivariant* quasi-conformal is induced by isometries.

We recall

Definition. quasi-isometric embedding

Let X, Y are metric spaces and $k \geq 0$. Then

$$f : X \to Y$$

is called a **k-quasi isometric embedding** if

$$-k + \frac{1}{k} d(x, x') \leq d(f(x), f(x')) \leq k + d(x, x')$$

for all $x, x' \in X$.

(Milnor-Schwarz) **Let X be a length metric space and $\Gamma \curvearrowright X$ acts by isometries properly such that X/Γ is compact then Γ is finitely generated and**

$$\begin{aligned} \Gamma &\to X \\ g &\mapsto gx \end{aligned}$$

is a quasi-isometry

This means

$$\begin{aligned} \Gamma_i &\cong_{\text{quasi}} \mathbb{R}H^3 \\ \gamma &\mapsto \gamma x \end{aligned}$$

so this induces

$$\begin{array}{ccc} \Gamma_1 & \xrightarrow{\varphi} & \Gamma_2 \\ {\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle q_2} \\ H^3 & \dashrightarrow & H^3 \end{array}$$

a quasi isometry

$$f : H^3 \to H^3$$

Definition. quasi geodesics

A curve

$$\gamma : [a, b] \to H^3$$

is called a **(k, ϵ)-quasi geodesic** if $\forall s \in [a, b]$

$$\frac{1}{k}|s - t| - \epsilon \leq d(\gamma(s), \gamma(t)) \leq k\,|s - t| + \epsilon$$

and put $b = \infty$ to get a geodesic ray.

The (k, ϵ)-quasi-isometries take geodesics to (k, ϵ)-quasi geodesics.

(Morse lemma) There exists a $R(k, \epsilon) > 0$ such that for every k, ϵ-quasi geodesic γ the geodesic A_γ with endpoints same as γ satisfies

$$\gamma \subset N_R(A_\gamma)$$

and

$$A_\gamma \subset N_R(\gamma)$$

Definition. Inducing map to the boundary from a quasi-isometry

Fix $x_0 \in H^3$

$$\partial f : \partial H^3 \to \partial H^3$$

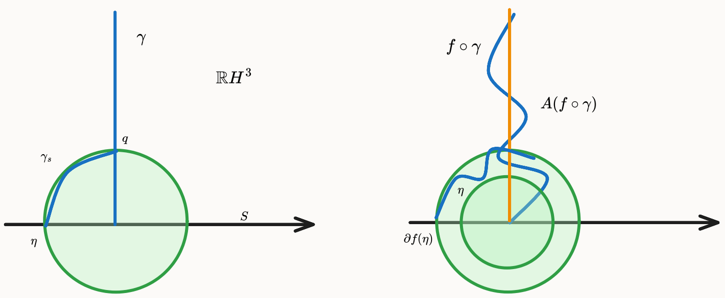
such that $s \in H^3$ there exists unique geodesic ray starting from x_0 to s then consider the geodesic $A(f \circ \gamma)$ and define

$$\partial f(s) := \lim_{t \to \infty} A(f \circ \gamma)(t)$$

Proposition: ∂f is a bijection.

As f is invertible, we have a quasi-isometry f^{-1} which induces an inverse of ∂f^{-1}

Proposition: Let $f : H^3 \to H^3$ be a (k, ϵ)-quasi-isometry. There exists $B(k, \epsilon) > 0$



such that for all geodesic rays γ and any hyperplane S perpendicular to γ

$$\mathrm{diam}(\pi_{A(f\circ\gamma)}(f(S))) < S$$

Let γ_s be the geodesic ray in S from q to s with endpoint η

- ...

Proposition: ∂f is continuous for a quasi-isometry $f : H^3 \to H^3$

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1. https://people.math.osu.edu/george.924/Ehresmann%20Theorem ↩
2. A counterexample Rigidity of compact complex manifolds ↩