

This note, that [was found here](#), as a part of [a collection](#) is written (completely with human hands) by [Rupadarshi Ray](#), was created on July 9, 2025 1:56:02 AM, was last modified on February 6, 2026 2:39:06 PM, and was exported on September 7, 2026 3:31:11 PM.

Orbits of smooth Lie group actions

Let $\theta : G \curvearrowright M$ be a smooth left Lie group action on a smooth manifold M and $p \in M$. Then the orbit map

$$\begin{aligned}\theta_{(p)} : G &\rightarrow M \\ g &\mapsto gp\end{aligned}$$

is smooth and has constant rank, so the isotropy group $G_p = \theta_{(p)}^{-1}(p)$ is a **closed subgroup** of G . Then

$$\theta_{(p)} : \frac{G}{G_p} \hookrightarrow M$$

is an injective immersion making the image, that is, the orbit $\theta^G(p)$ is an **immersed submanifold**

$$\iota : (G \setminus \{p\}, \theta_{(p)*} \mathcal{T}_{G/G_p}, \theta_{(p)*} \mathcal{A}_{G/G_p}) \hookrightarrow M$$

Moreover, if the action θ is *proper* then the orbit map $\theta_{(p)}$ is a proper smooth embedding (so in particular the orbit $\theta^G(p)$ is closed/properly embedded) and each stabilizer G_p is compact.

Proposition 7.26 (Properties of the Orbit Map). *Suppose θ is a smooth left action of a Lie group G on a smooth manifold M . For each $p \in M$, the orbit map $\theta^{(p)} : G \rightarrow M$ is smooth and has constant rank, so the isotropy group $G_p = (\theta^{(p)})^{-1}(p)$ is a properly embedded Lie subgroup of G . If $G_p = \{e\}$, then $\theta^{(p)}$ is an injective smooth immersion, so the orbit $G \cdot p$ is an immersed submanifold of M .*

Proposition 21.7 (Orbits of Proper Actions). *Suppose θ is a proper smooth action of a Lie group G on a smooth manifold M . For any point $p \in M$, the orbit map $\theta^{(p)} : G \rightarrow M$ is a proper map, and thus the orbit $G \cdot p = \theta^{(p)}(G)$ is closed in M . If in addition $G_p = \{e\}$, then $\theta^{(p)}$ is a smooth embedding, and the orbit is a properly embedded submanifold.*

21-17. Suppose a Lie group acts smoothly (but not necessarily properly or freely) on a smooth manifold M . Show that each orbit is an immersed submanifold of M , which is embedded if the action is proper.

	orbit $G \setminus \{p\}$	$G_p \leq G$	M/G
(general)	immersed submanifold of M	closed	(bad)
proper	closed submanifold of M , diffeomorphic to G/G_p	compact	Hausdorff
G is compact $\implies$ action is proper	compact submanifold of M , diffeomorphic to G/G_p	compact	Hausdorff
free		trivial	
proper, free		trivial	smooth manifold
transitive	orbit is M , diffeomorphic to G/G_p		point (smooth manifold)

action	orbit $G \setminus \{p\}$	
not proper	embedded but not closed	$\mathbb{R}_{>0} \curvearrowright \mathbb{R}$ whose orbits other than $\{0\}$ are $(-\infty, 0), (0, \infty)$ which are <i>not closed</i>
not proper	closed and embedded	
not proper	not embedded, closed	impossible?
not proper	not embedded, not closed	

If an orbit is closed, then it is an embedded submanifold?
[1]

Current note has 0 direct children and 0 total descendants.

- structures based on sets
 - manifolds
 - smooth $\mathbb{R}$ -manifold
 - Smooth Lie group action on a smooth manifold
 - Orbits of smooth Lie group actions

And it has 5 siblings.

- structures based on sets
 - manifolds

- smooth $\mathbb{R}$ -manifold
 - Smooth Lie group action on a smooth manifold
 - Smooth action by a compact Lie group
 - Generators of a Lie group action
 - Orbits of smooth Lie group actions
 - Proper Lie group actions
 - invariant Riemannian metric in a proper Lie group action

1. <https://mathoverflow.net/a/418306/562926> ↩