Info

This note, that was found here, as a part of a collection
is written (completely with human hands) by Rupadarshi Ray,
was created on July 5, 2025 2:56:50 PM,
was last modified on June 18, 2026 5:27:13 PM, and
was exported on September 7, 2026 3:31:16 PM.

Adjoint representation of a Lie group

Consider the conjugation action

Definition.

$$\begin{aligned}\mathfrak{c} : G &\to \mathsf{AutLie}_{\mathbb{R}}\mathsf{Grp}\\ x &\mapsto \mathfrak{c}_x(g) := xgx^{-1}\end{aligned}$$

and observe

$$\mathfrak{c}_x(i) = xix^{-1} = x$$

thus the action fixes $i \in G$.

Thus

$$\mathrm{d}_i\mathfrak{c}_x : \mathfrak{g} \to \mathfrak{g}$$

are linear maps. These are Lie algebra homomorphisms because $\mathfrak{c}_x$ are group homomorphisms.

This gives the linear action

Definition.

$$\begin{aligned}\mathrm{Ad} : G &\to \mathsf{AutLie}_{\mathbb{R}}(\mathfrak{g})\\ x &\mapsto \mathrm{Ad}_x := \mathrm{d}_i\mathfrak{c}_x\end{aligned}$$

equivariant with conjugation

By definition, for the 1-parameter subgroup

$$t \mapsto x\exp(tY)x^{-1}$$

we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} x\exp(tY)x^{-1} = \mathrm{d}_i\mathfrak{c}_x(Y) = \mathrm{Ad}_x(Y)$$

thus

$$\exp(\mathrm{Ad}_x Y) = \mathfrak{c}_x(\exp Y)$$

giving us the commutative

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "\exp"] \arrow[d, "\mathrm{Ad}_x"'] & G \arrow[d, "\mathfrak{c}_x"] \\
\mathfrak{g} \arrow[r, "\exp"'] & G
\end{tikzcd}$$

ad

We have the *derivative of the group action*

The derivative of Adjoint

Definition.

$$\begin{aligned}\mathrm{Ad} : G &\to \mathsf{AutLie}_{\mathbb{R}}(\mathfrak{g})\\ x &\mapsto \mathrm{Ad}_x := \mathrm{d}_i\mathfrak{c}_x\end{aligned}$$

is the adjoint action

$$\begin{aligned}\mathrm{ad} : \mathfrak{g} &\to \mathrm{Der}(\mathfrak{g})\\ X &\mapsto [X, -]\end{aligned}$$

by the Lie algebra onto itself

$$\mathfrak{D}_i\mathrm{Ad} = \mathrm{ad} : \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$$

In particular,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Ad}_{\exp(tX)} = \mathrm{ad}_X$$

thus

$$\exp(\mathrm{ad}_X) = \mathrm{Ad}_{\exp(X)}$$

where the first exponential the one on $\mathfrak{gl}(\mathfrak{g}) \to \mathsf{AutVec}_{\mathbb{R}}(\mathfrak{g})$.

$$\begin{tikzcd}
& & \mathfrak{g} \arrow[r, "\exp"] \arrow[d, "\mathrm{Ad}_x"', ""{name=A}] & G \arrow[d, "\mathfrak{c}_x"] \\
\mathfrak{g} \arrow[d, "\mathrm{ad}_x"', ""{name=B}] & & \mathfrak{g} \arrow[r, "\exp"'] & G \\
\mathfrak{g}
\arrow[from=B, to=A, "\exp"]
\end{tikzcd}$$

Current note has 2 direct children and 2 total descendants.

- structures based on sets
  - manifolds
    - $\mathbb{R}$-manifold group (Lie groups)
      - Adjoint representation of a Lie group
        - Ad^* orbits are symplectic manifolds
        - Adjoint group of a Lie group

And it has 18 siblings.

- structures based on sets
    - manifolds
        - $\mathbb{R}$-manifold group (Lie groups)
            - Adjoint representation of a Lie group
            - Conjugation action of a Lie group on itself
            - Classification of Lie groups
            - Compact Lie groups
            - Connected Lie groups with discrete center
            - Connected Lie groups with non-surjective exponential map
            - Discrete subgroups of Lie groups
            - Smooth maps onto a Lie group
            - Lie groups with surjective exponential map
            - Maurer-Cartan form
            - Left-invariant Lagrangian on a Lie group
            - Lie algebra of a lie group
            - Bi-invariant metrics on Lie groups
            - Left invariant Riemannian metric on a Lie group
            - Semi-simple $\mathbb{R}$-Lie groups
            - Almost simple groups
            - Virtual subgroups of Lie groups
            - Lie group with dilations