Info

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Critical values of smooth functions to $\mathbb{R}$

Definition. Critical points and critical values

Let $f \in \mathcal{C}^\infty(M)$. Then $p \in M$ is a **critical point** of f if $\mathrm{d}_p f = 0$. The real number $f(p)$ is called a **critical value** of f.

Definition. Hessian at critical points

Let $f \in \mathcal{C}^\infty(M)$ with a critical point $p \in M$, that is, $\mathrm{d}_p f = 0$. For $v, w \in T_pM$ with extensions $\tilde{v}, \tilde{w}$ to vector fields. We define

$$H_p f(v, w) := \tilde{v}_p(\tilde{w} f)$$

Proposition: The Hessian

Definition. Hessian at critical points

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is a well-defined, symmetric bilinear form on T_pM.

	$H_p(v,w)$	$v^\mathsf{T}\left[\frac{\partial^2 f}{\partial x_i \partial x_j}(p)\right]w$
nullity of f at p	$\dim\{v \in T_pM \mid H_p(v,w) = 0\}$ $= \dim\ker H_p(-,w)$	
p is a **non-degenerate** critical point	H_p is non-degenerate $\iff$ nullity at $p = 0$	$\left[\frac{\partial^2 f}{\partial x_i \partial x_j}(p)\right]$ is non-singular
index of f at p	maximal dimension of subspace of T_pM on which H	

Definition. A function $f \in \mathcal{C}^\infty(M)$ is a **Morse function** if all its critical points are non-degenerate.

Proposition:

Lemma 2.1. *Let f be a C^∞ function in a convex neighborhood V of 0 in $\mathbf{R}^n$, with $f(0) = 0$. Then*

$$f(x_1, \ldots, x_n) = \sum_{i=1}^n x_i g_i(x_1, \ldots, x_n),$$

for some suitable C^∞ functions g_i defined in V, with $g_i(o) = \frac{\partial f}{\partial x_i}(0)$.

Proof.

$$f(x_1, \ldots, x_n) = \int_0^1 \frac{\mathrm{d}f(tx_1, \ldots, tx_n)}{\mathrm{d}t}\mathrm{d}t = \int_0^1 \sum_{i=1}^n \frac{\partial f}{\partial x_i}(tx_1, \ldots, tx_n) \cdot x_i \mathrm{d}t.$$

Therefore we can let $g_i(x_1, \ldots, x_n) = \int_0^1 \partial f / \partial x_i (tx_1, \ldots, tx_n)\mathrm{d}t$. □

(Lemma of Morse) Let p be a *non-degenerate* critical point of f. Then there is a local coordinate $(y_1, \ldots, y_n) : U_p \to \mathbb{R}^n$ which $y_i(p) = 0$ such that

$$f = f(p) - \sum_{i=1}^{\mathrm{index}_p(f)} y_i^2 + \sum_i^n y_i^2$$

Corollary of

(Lemma of Morse) Let p be a *non-degenerate* critical point of f. Then there is a local coordinate $(y_1, \ldots, y_n) : U_p \to \mathbb{R}^n$ which $y_i(p) = 0$ such that

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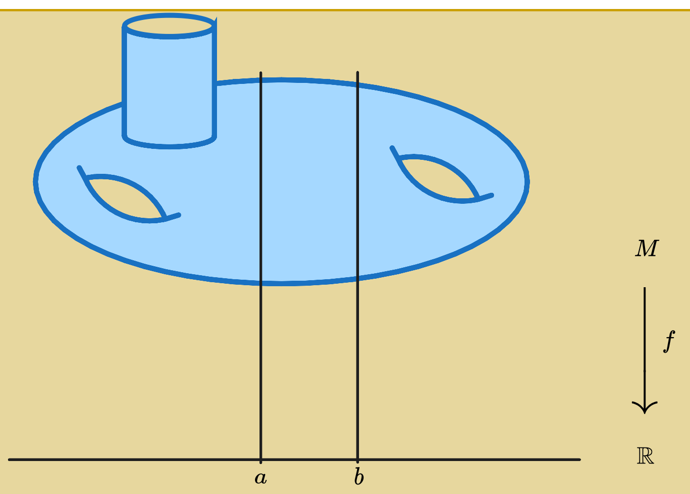
Non-degenerate critical points are isolated.

sublevel sets when we do not cross a critical value

Proposition: If a is a regular value of $f \in \mathcal{C}^\infty(M)$ then $f^{-1}(a)$ is a (closed) smooth submanifold of M and $f \leq a$ is a smooth manifold with boundary.

Let $f \in \mathcal{C}^\infty(M)$ for a smooth manifold M. Let $a < b$ be such that $f^{-1}[a, b]$ is *compact* and contains *no critical points* of f.



Then $f \leq a$ **is diffeomorphic to** $f \leq b$. Furthermore, $f \leq a$ is a deformation retract of $f \leq b$ so that the inclusion map

$$\iota : f \leq a \to f \leq b$$

is a homotopy equivalence.

- Choose a Riemannian metric g on M and consider the smooth vector field $\operatorname{grad}(f)$.
  - The vector field $\operatorname{grad}(f)$ vanishes precisely at critical points of f, so $\|\operatorname{grad}(f)\|_g \neq 0$ on $f^{-1}[a,b]$.
  - Let $\rho \in \mathcal{C}^\infty(M)$ such that

$$\rho\Big|_{f^{-1}[a,b]} = \frac{1}{\|\operatorname{grad}(f)\|_g^2}$$

    and vanishes outside of a closure of a pre-compact open neighborhood of $f^{-1}[a,b]$. Then $X := \rho\operatorname{grad}(f)$ has compact support.
  - By

    > A **compactly** supported smooth vector field on a smooth manifold is **complete**.

    we have its flow

$$\exp(tX) : M \to M$$

  - For $\exp(tX)q \in f^{-1}[a,b]$,

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\exp(tX)q) = \langle X, \operatorname{grad}(f)\rangle = 1$$

    which means

$$f(\exp(tX)q) = t + f(q)$$

  - Consider the diffeomorphism

$$\exp((b-a)X) : M \to M$$

    - If $q \in f^{-1}(a) \iff f(q) = a$ then

$$f(\exp((b-a)X)q) = b - a + f(q) = b$$

    - If $\exp((b-a)X)q \in f^{-1}[a,b]$ then

$$\begin{aligned} f(\exp((b-a)X)q) &= b - a + f(q) \\ \implies f(q) + b - a &\in [a,b] \\ \implies f(q) &\leq a \end{aligned}$$

    - Outside support of X, the diffeomorphism is identity. And preimage of $f^{-1}[a,b]$ is in $f \leq a$.
    - Thus the diffeomorphism maps $f \leq a$ to $f \leq b$.
  - Consider

$$\begin{aligned} r_t &: f \leq b \to f \leq a \\ r_t(q) &= \begin{cases} q & \text{on } f \leq a \\ \exp(t(a - f(q))X)(q) & \text{on } f^{-1}[a,b] \end{cases} \end{aligned}$$

    where $r_0 = \iota$ and r_1 is a retraction.
    - Hence $f \leq a$ is a deformation retract of $f \leq b$.

sublevel sets when we cross a critical value of *only one* non-degenerate critical point

Theorem 3.2. *Let $f\colon M \to \mathbf{R}$ be a smooth function, and let p be a non-degenerate critical point with index λ. Setting $f(p) = c$, suppose that $f^{-1}[c-\varepsilon, c+\varepsilon]$ is compact, and contains no critical point of f other then p, for some $\varepsilon > 0$. Then, for all sufficiently small ε, the set $M^{c+\varepsilon}$ has the homotopy type of $M^{c-\varepsilon}$ with a λ-cell attached.*

Morse homology

Current note has 1 direct children and 1 total descendants.

- structures based on sets
  - manifolds
    - smooth $\mathbb{R}$-manifold
      - Functions from smooth manifolds to $\mathbb{R}$
        - Critical values of smooth functions to $\mathbb{R}$
          - Smooth function on a smooth manifold with two critical points

And it has 3 siblings.

- structures based on sets
  - manifolds
    - smooth $\mathbb{R}$-manifold
      - Functions from smooth manifolds to $\mathbb{R}$
        - Critical values of smooth functions to $\mathbb{R}$
        - Equations for functions on smooth manifolds

- $\mathcal{C}^\infty(M)$