Info

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Lie bracket of two smooth vector fields on manifold

from $\mathrm{EndRing}(\mathcal{C}^\infty(M))$

Definition. $[X,Y]$

Given $X, Y \in \Gamma(TM)$, obtain the **Lie bracket of X and Y**

$$[X,Y] : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$$

defined by

$$[X,Y](f) := XYf - YXf$$

$$\exists ! [X,Y] \in \Gamma(TM)$$

Thus we obtain an operation $[\ ,\ ] : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$

$$(X,Y) \mapsto [X,Y]$$

$(\Gamma(TM), +, \ , [\ ,\ ])$ is a $\mathbb{R}$-Lie algebra and

$$[fX, gY] = fg[X,Y] + (fXg)Y - (gYf)X$$

from $\mathrm{LDiff}(M)$

- $$\begin{aligned}\mathrm{LDiff}(M) &\to \mathrm{LDiff}(M)\\ h &\xmapsto{\mathrm{conj}_g} ghg^{-1}\end{aligned}$$

  is a group homomorphism

- $$\mathrm{conj} : \mathrm{LDiff}(M) \curvearrowright \mathrm{LDiff}(M)$$

  is a group action by homomorphism

- $$\mathrm{Ad} : \mathrm{LDiff}(M) \curvearrowright \mathrm{Vec}(M)$$

  at g be the derivative of $\mathrm{conj}(g)$ given by

  $$\mathrm{Ad}(g)(Y) = \mathrm{conj}_{g*}(Y) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} g \exp(tY) g^{-1}$$

- We take another derivative producing $\mathrm{ad} : \mathrm{Vec}(M) \curvearrowright \mathrm{Vec}(M)$

  $$\mathrm{ad}(X)(Y)? = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \left.\frac{\partial}{\partial t}\right|_{t=0} \exp(sX)\exp(tY)\exp(-sX)$$

- $$\mathrm{ad}(X)(Y)(p)? = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \mathrm{conj}_{\exp sX\,*}(Y_p)$$

- assuming partial derivatives commute

  $$\begin{aligned}\left.\frac{\partial}{\partial s}\right|_{s=0} &\exp(sX)\exp(tY)\exp(-sX)(p)\\ &=? \exp(tY)_*(-X_p)\end{aligned}$$

  > - $$\mathrm{ad}(X)(Y) =? \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \exp(-sY)_*(X)$$
  >
  >   which is the Lie derivative
  >
  > $$\mathcal{L}_Y X = [Y, X]$$

$$\mathrm{ad}(X)(Y)? = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(tX)\exp(tY)\exp(-tX)\exp(-tY)$$

ways to handle the minus sign

Bug

The usual definition of commutator $[X,Y] := XY - YX$ as derivations of $\mathcal{C}^\infty(M)$ and infinitesimal generators of Lie group actions

$$V \mapsto \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(tV)(-)$$

produces an **anti-homomorphism** $\mathfrak{g} \to \mathrm{Vec}(M)$. This is expected because $\mathrm{LDiff}(M)$ acts on $\mathcal{C}^\infty(M)$ on the right.

- settle with the $[X,Y] = XY - YX$ brackets
  - does it agree with brackets in EndVec?
  - settle with the anti-homomorphism [1]
  - use $V \mapsto \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(-tV)(-)$ as definition of infinitesimal generators
    - this seems to produce $-X$ as the infinitesimal generator of flow of X
- use the brackets $-[X,Y]$, possibly defined using flows/adjoint action

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Current note has 0 direct children and 0 total descendants.

- structures based on sets
    - manifolds
        - smooth $\mathbb{R}$-manifold
            - Tangents on a smooth manifold
                - Tangent space of a smooth manifold
                    - Tangent vector fields on a smooth manifold
                        - Lie bracket of two smooth vector fields on manifold

And it has 4 siblings.

- structures based on sets
    - manifolds
        - smooth $\mathbb{R}$-manifold
            - Tangents on a smooth manifold
                - Tangent space of a smooth manifold
                    - Tangent vector fields on a smooth manifold
                        - Vector field on smooth manifold M as derivation of $\mathcal{C}^\infty(M)$
                        - Exponential of vector fields
                        - Index of a vector field
                        - Lie bracket of two smooth vector fields on manifold

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1. John M Lee - Introduction to smooth manifolds-Springer (2013).pdf ↩