Connected Lie groups with discrete center

> **Proposition:?** A connected Lie group G has **discrete center** $Z(G) \iff \mathfrak{g} := \mathrm{Lie}(G)$ has **trivial center** $\mathfrak{z}(\mathfrak{g}) = \{0\}$.

Let G be the **simply connected** Lie group with Lie algebra $\mathfrak{g}$ with **trivial center**. Then $Z(G)$ is the **maximal** discrete normal subgroup of G and $G/Z(G)$ the **minimal** quotient by a discrete normal subgroup.

$$
\begin{tikzcd}
& \{i_G\} \arrow[dl, dash, dashed] \arrow[dr, dash, dashed] & \\
N_1 \arrow[ddr, hook] & & N_2 \arrow[d, hook] \\
& & N_3 \arrow[dl, hook] \\
& Z(G) &
\end{tikzcd}
\qquad
\begin{tikzcd}
& G \arrow[dl, dash, dashed] \arrow[dr, dash, dashed] & \\
G/N_1 \arrow[ddr] & & G/N_2 \arrow[d] \\
& & G/N_3 \arrow[dl] \\
& G/Z(G) &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& & & G \arrow[dl, two heads] \arrow[dr, two heads] & \\
\mathfrak{Z}(\mathfrak{g}) = 0 & \mathfrak{g} \arrow[urr, bend left, "\exp"] \arrow[r, bend left, "\exp"] \arrow[urrr, "\exp"] \arrow[drrr, "\exp"] \arrow[ddrr, bend right, "\exp"] \arrow[ddd, "\mathrm{ad}"] & G/N_1 \arrow[ddr, two heads] & & G/N_2 \arrow[d, two heads] \\
& & & & G/N_3 \arrow[dl, two heads] \\
& & & \mathrm{Ad}(G) := \frac{G}{Z(G)} \arrow[d, hook, "\mathrm{Ad}"] & \\
& \mathfrak{gl}(\mathfrak{g}) \arrow[rr, "\exp"] & & GL(\mathfrak{g}) &
\end{tikzcd}
$$
