Info

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Proper actions by topological groups on topological spaces

Definition. Proper actions by topological groups on topological spaces

Let G be a topological group acting on a topological space X continuously

$$\theta : G \curvearrowright X$$

Then the action θ is called **proper** if the map

$$\begin{aligned} \theta \times \pi_2 : G \times X &\to X \times X \\ (g, p) &\mapsto (\theta^g(p), p) \end{aligned}$$

is a *proper map*. [1]

- Let $A \times B \subseteq X \times X$, then

$$(\theta \times \pi_2)^{-1}(A \times B) = \{(g, m) \in G \times X | \theta^g(m) \in A, m \in B\}$$

  - Now its projection onto G is

$$\pi_1((\theta \times \pi_2)^{-1}(A \times B)) = \left\{ g \in G \,\middle|\, \underbrace{\exists m \in B : \theta^g(m) \in A}_{\iff B \cap \theta^{g^{-1}}(A) \neq \emptyset} \right\}$$

  - For $A = B = K$ we have

$$\pi_1((\theta \times \pi_2)^{-1}(K \times K)) = \underbrace{\left\{ g \in G \,\middle|\, \underbrace{K \cap \theta^{g^{-1}}(K) \neq \emptyset}_{\iff \theta^g(K) \cap K \neq \emptyset} \right\}}_{=:G_K}$$

- Let

$$\theta : G \curvearrowright X$$

be a continuous action.

  - Let θ is a **proper** action. Then for any compact $K \subseteq X$ we have

$$G_K = \pi_1(\underbrace{(\theta \times \pi_2)^{-1}(K \times K)}_{\text{compact}})$$

    Thus G_K is compact.

  - Now let X is **Hausdorff** and for every compact $K \subseteq X$. Any compact subset $L \subseteq X \times X$ we have a compact set $K := \pi_1(L) \cup \pi_2(L)$ such that

$$\begin{gathered} L \subseteq K \times K \\ \implies \underbrace{(\theta \times \pi_2)^{-1}(L)}_{\text{closed}} \subseteq (\theta \times \pi_2)^{-1}(K \times K) \subseteq \underbrace{G_K \times K}_{\text{compact}} \end{gathered}$$

  - Thus by

    > **Proposition:** A closed subset of a compact space is compact.

    $(\theta \times \pi_2)^{-1}(L)$ is compact for every compact $L \subseteq X \times X$.

  - Thus $\theta \times \pi_2$ is a proper map $\implies$ θ is a proper action.

Hence,

> Let G be a group acting on a Hausdorff space X continuously
>
> $$\theta : G \curvearrowright X$$
>
> Then the action θ is called **proper** $\iff$ for every compact set $K \subseteq X$
>
> $$\begin{aligned} G_K &:= \pi_1((\theta \times \pi_2)^{-1}(K \times K)) \\ &= \{g \in G | \theta^g(K) \cap K \neq \emptyset\} \subseteq G \end{aligned}$$
>
> is compact.

[2]

Current note has 0 direct children and 0 total descendants.

- structures based on sets
  - Topological spaces
    - Group action on topological spaces
      - Proper actions by topological groups on topological spaces

And it has 3 siblings.

- structures based on sets
  - Topological spaces
    - Group action on topological spaces
      - Covering space action
      - Discontinuous group actions on topological spaces
      - Proper actions by topological groups on topological spaces

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1. **Definition.** Proper, perfect, universally closed maps

A continuous map $f : X \to Y$ between topological spaces is

- **proper** if preimage of compact sets $K \subseteq Y$, $f^{-1}(K)$ are compact in X
- **Bourbaki-proper** if for any topological space Z the product map

$$\mathrm{Id}_Z \times f : Z \times X \to Z \times Y$$

  is closed
- **perfect** if f is closed and has compact fibres $(f^{-1}(y)$ is compact for all $y \in Y)$
- **universally closed** if for every continuous map $g : Z \to Y$ the pullback

$$g \times_Y f : Z \times_Y X \to Z$$

  is closed
- **sequential escaper** if for any sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ *escaping to infinity* then $\{f(x_n)\}_{n\in\mathbb{N}}$ *escapes to infinity*

↩

2. John M. Lee.Introduction to Topological Manifolds-Springer-Verlag New York (2011), p.319 ↩