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Smooth function on a smooth manifold with two critical points

If M is a *compact* smooth manifold and $f \in \mathcal{C}^\infty(M)$ (must have at least two critical points) with **only two critical points**, then M **is homeomorphic to a sphere**, however in general, this may not be a diffeomorphism.

non-degenerate case

Proof. This follows from Theorem 3.1 together with the Lemma of Morse (Lemma 8.3). The two critical points must be the minimum and maximum points. Say that $f(p) = 0$ is the minimum and $f(q) = 1$ is the maximum. If ε is small enough then the sets $M^\varepsilon = f^{-1}[0, \varepsilon]$ and $f^{-1}[1 - \varepsilon, 1]$ are closed n-cells by Lemma 8.3. But M^ε is homeomorphic to $M^{1-\varepsilon}$ by Theorem 3.1. Thus M is the union of two closed n-cells, $M^{1-\varepsilon}$ and $f^{-1}[1 - \varepsilon, 1]$, matched along their common boundary. It is now easy to construct a homeomorphism between M and $\mathbb{S}^n$. □

degenerate case

Remark. The theorem remains true even if the critical points are degenerate. However, the proof is more difficult. (Compare Milnor, *Differential topology*, in [Mil64, Theorem 1]; or [Ros60, Lemma 1].)

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And it has 1 siblings.

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