Info

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$GL(2)(\mathbb{Z}) \curvearrowright T^2$

#Wiki/group-action/discrete

Definition. Linear maps on the 2-torus

From the module homomorphism

$$(\mathsf{Mat}(2,\mathbb{Z}), \times) \to (\mathsf{EndMan}(T^2), \circ)$$

we have

$$\begin{aligned} GL(2)(\mathbb{Z}) &\curvearrowright T^2 \\ \begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix} &\overset{S}{\mapsto} \begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix}\end{aligned}$$

sub-actions

sub-action	property	MCG
$GL(2)(\mathbb{Z})$		$\pi_0(\text{Homeo}(T^2))$?
$SL(2)(\mathbb{Z})$	orientation preserving	$\pi_0(\text{Homeo}_+(T^2))$
$\Gamma < SL(2)(\mathbb{Z})$ torsion-free		

induced action on π_1

Proposition: From the group action

Definition. Linear maps on the 2-torus

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the induced map

$$GL(2)(\mathbb{Z}) \to \mathsf{AutGrp}(\underbrace{\pi_1(T^2, 0)}_{\mathbb{Z}^2}) \cong GL(2)(\mathbb{Z})$$

is the identity map.

We identify

$$\begin{aligned} \pi_1(T^2, 0) &\cong \mathbb{Z}^2 \subset \mathbb{R}^2 \\ \gamma &\mapsto \tilde{\gamma}(1)\end{aligned}$$

The transitive Affine group action

$$\begin{aligned} GL(2)(\mathbb{R}) \ltimes \mathbb{R}^2 &\curvearrowright \mathbb{R}^2 \\ x &\overset{(A,b)}{\longmapsto} Ax + b\end{aligned}$$

with stabilizer G_p. Making

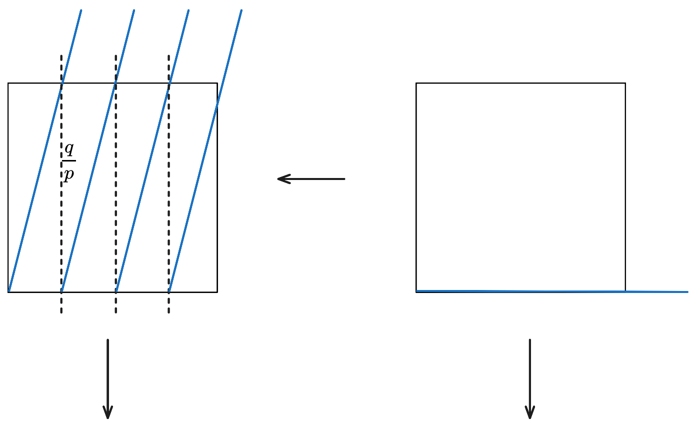
$$\frac{GL(2)(\mathbb{R}) \ltimes \mathbb{R}^2}{G_p} \cong \mathbb{R}^2$$

Now consider $\{I\} \times \mathbb{Z}^2 < GL(2)(\mathbb{R}) \ltimes \mathbb{R}^2 \curvearrowright \mathbb{R}^2$ and quotient $\mathbb{R}^2$ with it

$$\frac{GL(2)(\mathbb{R}) \ltimes \mathbb{R}^2/G_p}{\{I\} \times \mathbb{Z}^2} \cong \frac{\mathbb{R}^2}{\mathbb{Z}^2}$$

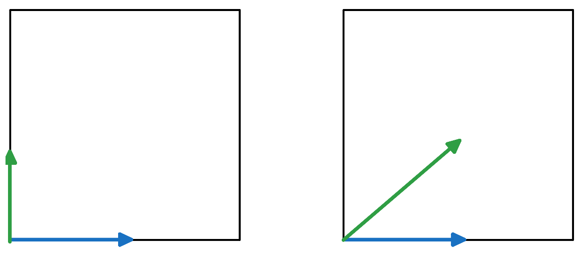
giving the action

$$\frac{GL(2)(\mathbb{Z}) \ltimes \mathbb{R}^2}{\{I\} \times \mathbb{Z}^2} \curvearrowright \frac{\mathbb{R}^2}{\mathbb{Z}^2}$$

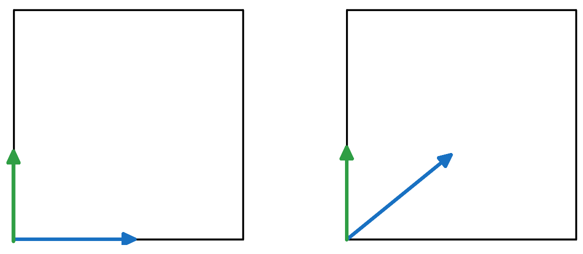


-
- takes (p,q) class to $(1,0)$?
- elements of mapping class group of torus?
- $$\begin{aligned} T_a : S^1 \times (0,1) &\to S^1 \times (0,1) \\ (\theta, r) &\mapsto (\theta + r, r)\end{aligned}$$
- $$\begin{aligned} T_{1,0} : S^1 \times S^1 &\to S^1 \times S^1 \\ (\theta_1, \theta_2) &\mapsto (\theta_1 + \theta_2, \theta_2)\end{aligned}$$

- $$(x, y) \mapsto (x + y, y)$$



$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

- $$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$
- $$(x, y) \mapsto (x + 2y, y)$$
- $$\begin{bmatrix} 1 & p \\ 0 & 1 \end{bmatrix}$$
- $$(x, y) \mapsto (x, x + y)$$

  is

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ q & 1 \end{bmatrix} = \begin{bmatrix} 1 + pq & p \\ q & 1 \end{bmatrix}$$

$$(x, y) \mapsto ((1 + pq)x + py, qx + y)$$

$$\begin{aligned} T_{p,q} : S^1 \times S^1 &\to S^1 \times S^1 \\ (\theta_1, \theta_2) &\mapsto (\theta_1 + p\theta_2, \theta_2 + q\theta_1) \end{aligned}$$

Current note has 3 direct children and 3 total descendants.

- stretchy
  - $GL(2)(\mathbb{Z}) \curvearrowright T^2$
    - Arnold cat map $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \curvearrowright T^2$
    - $SL(2)(\mathbb{Z}) \curvearrowright T^2$
    - Suspension flows of $SL(2)(\mathbb{Z}) \curvearrowright T^2$

And it has 62 siblings.

- stretchy
  - Trefoil 3_1 knot
  - The only compact connected Lie group that can act faithfully on a torus is itself
  - Hecke algebras
  - $\overline{B^n}$
  - $\mathrm{BS}(1, 2)$
  - $(\mathbb{C}, +, \times)$
  - $D^* := D \setminus \{0\}$
  - Cantor set
  - Configuration space of N rigid points in $\mathbb{R}^3$
  - Free groups
  - $GL(2)(\mathbb{Z}) \curvearrowright T^2$
  - H_2^3
  - $[a, b]$
  - Figure 8 knot
  - Lamplighter group on $\mathbb{Z}_2$
  - $\beta\mathbb{N}$
  - $\#_2\mathbb{R}P^2$
  - $\prod_{i \in \mathbb{N}} \{1, \dots, n_i\}$
  - $\mathbb{R}P^n$
  - $(\mathbb{Q}, +, \times, <)$
  - $\mathbb{Q}_{\hat{p}}$
  - $\overline{\mathbb{Q}}$
  - $\mathbb{Q}_{\text{constr}}$
  - $\mathbb{Q}(\xi_n)$
  - $\mathcal{O}_{\overline{\mathbb{Q}}}$
  - $(\mathbb{R}, +, \times, <, \sup)$
  - Homeomorphism of $\mathbb{R}^2$ that sends $\mathbb{N}$ to a countable closed discrete subset
  - $\mathbb{R}^2 \times S^1$
  - $\mathbb{R}^n$
  - $\mathbb{R} \times S^1$
  - $\mathbb{R} \times S^2$
  - S^1
  - $S^1 \wedge S^1$
  - $S^1 \times S^n$
  - S^2 and $\mathbb{C}\boldsymbol{P}^1$

- Mapping torus of the antipodal map of S^2
- S^3
- S^n
- I-bundle
- Symmetric group
- $\mathbb{T}$
- T^2
- T_g^2
- $T_{0,0,3}^2$
- $\mathbb{R}^2 \setminus \{0\}$
- Three $T_{1,0,1}^2$ glued at their boundaries
- $T_{1,1,0}^2$
- T_2^2
- $T_g^2 \times [0,1]$
- $\#_{\mathbb{N}} T^2$
- Thompson's group
- T^n
- 3-valent tree
- 4-valent tree
- $[0,1]^{\mathbb{N}}$ with product topology
- Whitehead manifolds
- $\mathbb{Z}$
- $\mathbb{Z}[\sqrt{m}]$
- $\mathbb{Z}[X]$
- $\mathbb{Z}^{\mathbb{N}}$
- $\mathbb{Z}[i]$
- $\mathbb{Z}[i\sqrt{5}]$