Info

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Laplace operator on $\mathbb{R}^n$

Definition. We have the Laplace operator

$$\Delta : \mathcal{C}^2 \to \mathcal{C}$$

$$\begin{aligned}\Delta &:= \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \\ &= \frac{1}{r^{n-1}} \frac{\partial}{\partial r}\left(r^{n-1} \frac{\partial}{\partial r}(-)\right) + \frac{1}{r^2}\Delta_{S^{n-1}} \\ &= \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{S^{n-1}}\end{aligned}$$

$$\begin{aligned}&\frac{\partial}{\partial r}(r^{n-1}v'(r)) = 0 \\ \implies &r^{n-1}v'(r) = a \\ \implies &v'(r) = \frac{a}{r^{n-1}} \\ \implies &v(r) = \begin{cases} a\log r + b & n = 2 \\ \dfrac{a}{r^{n-2}} + b & n \geq 3 \end{cases}\end{aligned}$$

Definition. The function

$$\Phi : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$$

$$\Phi(x) := \begin{cases} -\dfrac{1}{2\pi}\log|x| & n = 2 \\ \dfrac{1}{n(n-2)\alpha(n)} \dfrac{1}{|x|^{n-2}} & n \geq 3 \end{cases}$$

is the **fundamental solution of Laplace's equation** $\Delta u = 0$ **on** $\mathbb{R}^n$.

Proposition: For

> **Definition.** The function
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we have

$$\begin{aligned}|\mathfrak{D}_x\Phi| &\lesssim \frac{1}{|x|^{n-1}} \text{ for } x \neq 0 \\ |\mathfrak{D}_x^2\Phi| &\lesssim \frac{1}{|x|^n} \text{ for } x \neq 0\end{aligned}$$

convolution with fundamental solution solves Poisson's equation

Let $f \in \mathcal{C}_c^2(\mathbb{R}^n)$ then the convolution with the fundamental solution

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$$\begin{aligned}\Phi * f &: \mathbb{R}^n \to \mathbb{R} \\ x &\mapsto \int_{y\in\mathbb{R}^n} \Phi(x-y)f(y)\end{aligned}$$

is $\Phi * f \in \mathcal{C}^2(\mathbb{R}^n)$ and solves the Poisson's equation $-\Delta u = f$ in $\mathbb{R}^n$.

Let $f \in \mathcal{C}_c^2(\mathbb{R}^n)$ and φ be a fundamental solution of Δ on $\mathbb{R}^n$. Then for

$$u := \varphi * f$$

we have

$$\begin{aligned}\Delta u(x) &= \int_{\mathbb{R}^n} \varphi(y)\Delta f(x-y)\mathrm{d}y \\ &= \underbrace{\int_{B_\epsilon(0)} \varphi(y)\Delta f(x-y)\mathrm{d}y}_{\xrightarrow{\epsilon\to 0} 0} + \int_{\mathbb{R}^n\setminus B_\epsilon(0)} \varphi(y)\Delta f(x-y)\mathrm{d}y \\ &\xrightarrow{\epsilon\to 0} 0 - \int_{\mathbb{R}^n\setminus B_\epsilon(0)} \operatorname{grad}(\varphi)\cdot\operatorname{grad}(f)(x-y)\mathrm{d}y \\ &+ \underbrace{\int_{\partial B_\epsilon(0)} \varphi(y)\frac{\partial f}{\partial \hat{n}}(x-y)\mathrm{d}y}\end{aligned}$$

- Now

$$\int_{\partial B_\epsilon(0)} |\varphi(y)|\mathrm{d}y \leq \begin{cases} c\,|\log(\epsilon)|\epsilon \\ c\epsilon \end{cases} \xrightarrow{\epsilon\to 0} 0$$

- and

$$
\begin{aligned}
&-\int_{\mathbb{R}^n\setminus B_\epsilon(0)} \operatorname{grad}(\varphi)\cdot\operatorname{grad}(f)(x-y)\mathrm{d}y \\
&= \cancelto{0}{\int_{\mathbb{R}^n\setminus B_\epsilon(0)} f(x-y)\operatorname{grad}(\varphi)(y)\mathrm{d}y} \quad - \int_\partial \frac{\partial\varphi}{\partial\hat{n}} f(x-y)\mathrm{d}y \\
&= -\{ \\
&\xrightarrow{\epsilon\to 0} -f(x)
\end{aligned}
$$

Mean-value

Let $u\in\mathcal{C}^2(U)$ and is harmonic, then

$$
u(x)=\frac{1}{m(\partial B_r(x))}\int_{\partial B_r(x)} u=\frac{1}{m(B_r(x))}\int_{B_r(x)} u
$$

for each ball $B_r(x)\subseteq U$.

Let

$$
\begin{aligned}
\varphi(r) &:= \frac{1}{m(\partial B_r(x))}\int_{\partial B_r(x)} u \\
&= \frac{1}{m(\partial B_1(0))}\int_{y\in\partial B_1(0)} u(x+ry)
\end{aligned}
$$

- Then

$$
\begin{aligned}
\varphi'(r) &= \frac{1}{m(\partial B_1(0))}\int_{y\in\partial B_1(0)} \frac{\partial u}{\partial\hat{n}}(y) \\
&= \frac{1}{m(\partial B_1(0))}\int_{y\in\partial B_1(0)} \cancelto{0}{\Delta u} \\
&= 0
\end{aligned}
$$

Estimates on derivatives

Let u is harmonic on $U\subseteq\mathbb{R}^n$. Then

$$
\left|\mathfrak{D}_p^\alpha u\right| \le \frac{c(k)}{r^{n+k}} \| u \|_{L^1(B_r(x_0))}
$$

Green's function

$$
\begin{aligned}
\Delta G(-,y) &= \delta_x \\
G|_{\partial U} &= 0
\end{aligned}
$$

Current note has 1 direct children and 1 total descendants.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $_\Gamma\backslash^{\mathbb{R}^n}, _\Gamma\backslash^{\mathbb{C}^n}$
    - Laplace operator on $\mathbb{R}^n$
      - Laplace operator on (flat) upper half space

And it has 39 siblings.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $_\Gamma\backslash^{\mathbb{R}^n}, _\Gamma\backslash^{\mathbb{C}^n}$
    - Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
    - Open subsets of $\mathbb{R}^n$ with C^2 boundary
    - Complex ODEs in $\mathbb{C}$
    - Circle packing on $\mathbb{R}^2$
    - Circle packing converges to the Riemann biholomorphism
    - Bounded connected open subsets of $\mathbb{C}^n$
    - Connected circular open subsets of $\mathbb{C}^n$
    - Continuous functions on $\mathbb{R}^d$
    - Dyadic cubes
    - Curves
    - Differentiable functions
    - Differential forms on $\mathbb{R}^n$
    - Fourier-Wigner transform
    - Harmonic functions composed with conformal maps
    - Hilbert transform
    - Fourier-Cauchy-Poisson correspondence of holomorphic and harmonic functions on the unit disk and their boundary values
    - Holomorphic subsets of $\mathbb{C}^n$
    - Regular hypersurfaces in $\mathbb{R}^{2n}$
    - Orientable hypersurfaces in $\mathbb{R}^n$
    - Laplace operator on $\mathbb{R}^n$
    - $l^\infty(\mathbb{R})$
    - Lebesgue measurable subsets of and functions on $\mathbb{R}^n, T^n, S^n$
    - Lebesgue measure of boundary of open sets in $\mathbb{R}^n$
    - Local $\mathbb{R}$-algebras
    - $l^p(\mathbb{R})$
    - Metric density of subsets of $\mathbb{R}^n$
    - Monotone functions on $\mathbb{R}$
    - Cauchy integral of periodic functions
    - Polygons with integer vertices
    - Open smooth maps $U\subseteq\mathbb{R}^2\to\mathbb{C}$
    - Discrete subgroups of $\mathbb{R}^n$

- Discrete cocompact subgroups of $\mathbb{R}^n$, flat tori
- Ramified germs of smooth and holomorphic functions
- Open subsets of $\mathbb{R}^n$
- Open subsets of $\mathbb{R}^n$ equipped with the flat metric
- Closed subsets of $\mathbb{R}^n$ equipped with smooth functions
- Quasi-analytic smooth functions on $\mathbb{R}$
- Star-shaped subsets of $\mathbb{R}^n$
- ODEs in $\mathbb{R}^n$ $\leftrightarrow$ Vector fields in $\mathbb{R}^n$