Info

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When is a local homeomorphism between topological spaces a closed map?

Local homeomorphisms, even covering maps, are not closed in general.

Consider the covering

$$\begin{aligned}\mathbb{R} &\to S^1\\ x &\mapsto \exp(2\pi i x)\end{aligned}$$

We know

$$f^{-1}(1) = \mathbb{Z}$$

and we have an evenly covered nb of $\mathbb{Z}$ mapping to 1

$$\bigsqcup_{m\in\mathbb{Z}} \left(\left(-\frac{1}{2}, \frac{1}{2}\right) + m\right) = f^{-1}(1)$$

Also note

$$\frac{1}{n+2} \xrightarrow{n\geq 1, n\to\infty} 0 \implies \exp\left(\frac{2\pi i}{n+2}\right) \xrightarrow{n\geq 1, n\to\infty} 1$$

Then, as this is an infinitely sheeted cover, we pick a preimage of each element of the converging sequence in *different* sheets

$$\frac{1}{|m|+2} + m \in \left(-\frac{1}{2}, \frac{1}{2}\right) + m$$

implying

$$\left\{\frac{1}{|m|+2} + m \middle| m \in \mathbb{Z}\right\}$$

is a discrete, closed set.

$$\underbrace{\left\{\frac{1}{|m|+2} + m \middle| m \in \mathbb{Z}\right\}}_{\text{closed}\subseteq\mathbb{R}} \mapsto \underbrace{\left\{\exp\left(\frac{2\pi i}{n+2}\right) \middle| n \in \mathbb{N}\right\}}_{\text{not closed}\subseteq S^1}$$

A local homeomorphism with one infinite fibre *maybe* a closed map.

Consider the constant map

$$\mathbb{N} \to \{0\}$$

Then it is a local homeomorphism and a closed map.

We may extend the proof for exp to a general case by excluding existence of isolated points in the domain:

Let X be a separable metric space with no isolated points and Y be Hausdorff and first countable. Consider a local homeomorphism

$$f : X \to Y$$

such that there is a $y \in Y$ such that $f^{-1}(y)$ is infinite. Such a map f is **not closed**.

- $f^{-1}(y)$ is a discrete infinite set.
  - As X is separable, any discrete infinite set is at most countable.
    - So $f^{-1}(y) = \{x_n | n \in \mathbb{N}\}$ where x_n are all distinct.
    - Pick $r_n > 0$ such that $B_{r_n}(x)$ are all pairwise disjoint and

      $$f : B_{r_n}(x) \to f(B_{r_n}(x))$$

      is a homeomorphism by

      > Let $S \subseteq X$ be such that there is a collection $\{U_a\}_{a\in S}$ of open sets such that for all $a \in S$
      >
      > $$U_a \cap S = \{a\}$$
      >
      > (which makes S a discrete subset). Then there is a pairwise disjoint collection $\{V_a\}$ of subsets $V_a \subseteq U_a$ such that $V_a \cap S = \{a\}$ (making S strongly discrete).
  - Then pick some

    $$z_n \in B_{r_n}(x_n) \cap f^{-1}(U_n)$$

    where $\{U_n\}$ is a local basis of y so that $\{z_n\}$ is discrete closed set and

    $$f(z_n) \xrightarrow{n\to\infty} y$$

    - Thus

      $$\underbrace{\{z_n\}}_{\text{closed}} \mapsto \underbrace{\{f(z_n)\}}_{\text{not closed}}$$

[1]

Current note has 0 direct children and 0 total descendants.

- structures based on sets
    - Topological spaces
        - map
            - When is a local homeomorphism between topological spaces a closed map?

And it has 12 siblings.

- structures based on sets
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            - When is a local homeomorphism between topological spaces a covering map
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1. https://math.stackexchange.com/a/4743954 ↩