Info

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Fourier transform on S^1, Fourier series on $[0,1]$

Definition. Fourier transform on S^1, Fourier series on $[0,1]$

Let $f \in L^1(S^1)$. Then for $n \in \mathbb{Z}$ the n**-th Fourier coefficient** of f is

$$\begin{aligned}\hat{f}(n) &:= \frac{1}{2\pi i}\int_{S^1}\frac{f(z)}{z^{n+1}}\mathrm{d}z \\ &= \frac{1}{2\pi i}\int_{[-\pi,\pi]} f(e^{i\theta})e^{-in\theta}\frac{\mathrm{d}(e^{i\theta})}{e^{i\theta}} \\ &= \frac{1}{2\pi}\int_{[-\pi,\pi]} f(e^{i\theta})e^{-in\theta}\mathrm{d}\theta\end{aligned}$$

and the $\mathbb{Z}$-sequence f is the **Fourier transform** of f.
Let $f \in L^1[0,1]$. Then for $n \in \mathbb{Z}$ the n**-th Fourier coefficient** of f is

$$\hat{f}(n) := \int_{x\in[0,1]} f(x)e^{-2\pi inx}$$

and the $\mathbb{Z}$-sequence $\hat{f}$ is the **Fourier series** of f.

- For $f \in L^1[0,1]$, the Fourier coefficients of f

  $$\hat{f}(n) := \int_{x\in[0,1]} f(x)e^{-2\pi inx}$$

  is bounded

  $$\begin{aligned}\left|\hat{f}(n)\right| &\le \left|\int_{x\in[0,1]} f(x)e^{-2\pi inx}\right| \\ &\le \int_{x\in[0,1]} |f(x)| \\ &= \|f\|_1\end{aligned}$$

  by its L^1-norm.

Actually, the Fourier coeffients converge to 0 at infinity:

(Riemann-Lebesgue lemma for Fourier transform on S^1) Let $f \in L^1(S^1)$ then the Fourier coefficients $\hat{f}(k)$ converge to 0 in both $\mathbb{Z}$-directions

$$\hat{f}(k), \hat{f}(-k) \xrightarrow{k\to\infty} 0$$

That is

$$\widehat{} : L^1[0,1] \to \mathcal{C}_0(\mathbb{Z},\mathbb{C})$$

This does not hold for finite measures.

Definition. Fourier transform of signed measures on S^1

Let $\mu \in \mathrm{Meas}(\mathfrak{B}_{[0,1]},\mathbb{R})$ be a finite $\mathbb{R}$-measure on $[0,1]$. Then

$$\hat{\mu}(n) := \int_{[0,1]} e^{-2\pi in\theta}\mathrm{d}\mu$$

The Fourier summation

Definition. N-th **partial Fourier summation** of f is the trignometric polynomial

$$(S_N f)(x) := \sum_{-N\le n\le N}\hat{f}(n)e^{2\pi inx}$$

- Let $f \in \mathcal{C}^{\mathsf{A}}[0,1]$ so that its differentiable almost everyehere and $f' \in L^1[0,1]$.
  - Then for all $n \in \mathbb{Z}\setminus\{0\}$ we have

    $$\begin{aligned}\widehat{f}(n) &= \int_{x\in[0,1]} f(x)e^{-2\pi inx} \\ &= \int_{x\in[0,1]} f(x)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{-2\pi in}\frac{\mathrm{d}}{\mathrm{d}x}e^{-2\pi inx}\right) \\ &= f(0) - f(1) - \frac{1}{-2\pi in}\int_{x\in[0,1]} f'(x)e^{-2\pi inx}\end{aligned}$$

  - If $f(0) = f(1)$, that is, $f \in \mathcal{C}^{\mathsf{A}}(S^1)$, then for all $n \in \mathbb{Z}\setminus\{0\}$ we have

    $$\widehat{f'}(n) = (2\pi i)n\hat{f}(n)$$

- If $f \in \mathcal{C}^{\mathsf{A}+1}[0,1]$ implying $f^{(2)} \in L^1[0,1]$ and

  $$f(0) = f(1), f'(0) = f'(1)$$

  then

  $$\widehat{f''}(n) = (2\pi i)n\hat{f'}(n) = (2\pi i)^2n^2\hat{f}(n)$$

- **Inductively,**

  $$\widehat{f^{(k)}}(n) = (2\pi in)^k\hat{f}(n)$$

Current note has 0 direct children and 0 total descendants.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $\Gamma\backslash^{\mathbb{R}^n}, \Gamma\backslash^{\mathbb{C}^n}$
    - Fourier
      - Fourier transform on S^1, Fourier series on $[0,1]$

And it has 10 siblings.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $\Gamma\backslash^{\mathbb{R}^n}, \Gamma\backslash^{\mathbb{C}^n}$
    - Fourier
      - Fourier transform on $L^2(-A,A) \leq L^2(\mathbb{R})$
      - $\frown : L^2(0,\infty) \cong_{\mathsf{Hilb}} \mathscr{O}^2(H^2_{\mathbb{U}})$
      - Fourier transform on $\mathbb{R}^n$
      - Fourier transform on S^1, Fourier series on $[0,1]$
      - Functions on S^1 with absolutely converging Fourier series, $\check{l}^1(S^1)$
      - Fourier transform of distributions on S^1
      - $\frown : L^1[0,1] \to \mathcal{C}_0(\mathbb{Z},\mathbb{C})$
      - $\frown : L^2[0,1] \cong_{\mathsf{Hilb}} l^2(\mathbb{Z},\mathbb{C})$
      - Fourier transform of measurable subsets
      - Unsharpness principles