

Info

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Regular representation of a locally compact group

The left translations

Definition. Left translation on a topological group

The **left translation** on G by $x \in G$ is the homeomorphism

$$\begin{aligned} L_x : G &\rightarrow G \\ a &\mapsto xa \end{aligned}$$

The **left translation action** of G is

$$\begin{aligned} L : G &\curvearrowright G \\ a &\mapsto L_x(a) = xa \end{aligned}$$

pullbacks functions by

$$L_x^* f := f \circ L_x = f(x-)$$

But, it is not a left action because

$$L_x^*(L_y^* f) = (f \circ L_y) \circ L_x = f \circ L_{yx}$$

So we define the left translation of functions by

Definition.

$$f \mapsto \tau_x(f) := L_{x^{-1}}^* f$$

Let G be a locally compact group with Haar measure ν , $1 \leq p < \infty$ and $f \in L^p(G, \nu)$. Then

$$\|\tau_x f - f\|_p \xrightarrow{x \rightarrow i_G} 0$$

and

$$\|R_x^* f - f\|_p \xrightarrow{x \rightarrow i_G} 0$$

Fix a compact nb V of i_G and $g \in \mathcal{C}_c(G)$

- Let

$$K := \text{supp} g V^{-1} \cup V \text{supp} g$$

which is compact and $\tau_y g$ and $R_y^* g$ is supported in K when $y \in V$.

- Hence,

$$\|\tau_y g - g\|_p \leq \nu(K)^{1/p} \|\tau_y g - g\|_\infty \xrightarrow{y \rightarrow i_G} 0$$

by

Let G be a topological group and $f \in \mathcal{C}_c(G)$. Then f is left and right uniformly continuous

A function $f : G \rightarrow \mathbb{C}$ is **left uniformly continuous** if

$$\|\tau_y f - f\|_\infty \xrightarrow{y \rightarrow i_G} 0$$

and **right uniformly continuous** if

$$\|R_y^* f - f\|_\infty \xrightarrow{y \rightarrow i_G} 0$$

- Now let $f \in L^p(G)$ where we have

$$\|\tau_y f\|_p = \|f\| = \Delta(y)^{1/p} \|R_y f\|_p$$

Given $\epsilon > 0$ we choose $g \in \mathcal{C}_c(G)$ such that $\|f - g\|_p \leq \epsilon$, so

$$\|R_y f\|_p \leq c \|f\|_p$$

for $y \in V$ and then

$$\begin{aligned} \|R_y f - f\|_p &\leq \|R_y(f - g)\|_p + \|R_y g - g\|_p + \|g - f\|_p \\ &\leq (c + 1) \|f - g\|_p + \|R_y g - g\|_p \\ &\leq (c + 1) \epsilon + \epsilon \end{aligned}$$

Let G be a locally compact group, ν be a Haar measure on G and $1 \leq p < \infty$. The **right-regular representation**

$$\begin{aligned} G &\rightarrow BU(L^p(G, \nu)) \\ (g\varphi)(x) &:= \varphi(xg) \end{aligned}$$

is a well-defined strongly continuous and unitary representation of G .

Current note has 0 direct children and 0 total descendants.

- [structures based on sets](#)

- Topological spaces
 - Topological groups
 - Locally compact Hausdorff topological group
 - Regular representation of a locally compact group

And it has 2 siblings.

- structures based on sets
 - Topological spaces
 - Topological groups
 - Locally compact Hausdorff topological group
 - Regular representation of a locally compact group
 - Almost invariant vectors of a unitary representation

1. <https://planetmath.org/compactlysupportedcontinuousfunctionsaredenseinlp> ↩