Info

This note, that was found here, as a part of a collection
is written (completely with human hands) by Rupadarshi Ray,
was created on February 21, 2024 9:52:02 PM,
was last modified on February 21, 2024 10:28:53 PM, and
was exported on September 7, 2026 3:24:53 PM.

Homotopy invariance of singular homology functor on topological spaces

Definition. Singular chain complex functor on category of topological spaces

We define the **singular chain complex functor** on category of topological spaces

$$\mathrm{sing}_* : \mathsf{Top} \to \mathsf{Ch}_*(\mathsf{Ab})$$

which

- maps topological spaces to the singular chain complex

$$X \mapsto \mathrm{sing}_*(X)$$

- maps continuous maps

$$f : X \to Y$$

to the morphism of chain complexes

$$\begin{aligned} f_* : \mathrm{sing}_*(X) &\to \mathrm{sing}_*(Y) \\ \sum n_i \sigma_i &\mapsto \sum (f \circ \sigma_i) \end{aligned}$$

is a functor.

states

prism chain homotopy operator

Let $f, g : X \to Y$ are continuous maps with homotopy

$$T : X \times I \to Y$$

from f to g and a singular simplex $\sigma : \Delta^n \to X$.

Using this we can define the prism operators

$$\begin{aligned} P : \mathrm{sing}_n(X) &\to \mathrm{sing}_{n+1}(X) \\ P(\sigma) &:= \sum_i (-1)^i \, T \circ (\sigma \times \mathsf{Id})\Big|_{[v_0,...v_i,w_i,...,w_n]} \end{aligned}$$

> Claim:

$$\partial P + P\partial = g_* - f_*$$

$$P(\sigma) := \sum_i (-1)^i \, T \circ (\sigma \times \mathsf{Id})\Big|_{[v_0,...v_i,w_i,...,w_n]}$$

So

$$\partial P(\sigma) = \sum_i (-1)^j (-1)^i \, T \circ (\sigma \times \mathsf{Id})\Big|_{[v_0,...v_i,w_i,...,w_n]}$$

homotopic maps induce same maps on homology

If $f, g : X \to Y$ are homotopic continuous maps, then

$$f_* = g_*$$

on $H_n \mathrm{sing}_*(X)$, implying

$$H_n \mathrm{sing}_* : \mathsf{hTop} \to \mathsf{Ab}$$

is a functor.

homotopy invariance

If $f : X \to Y$ is a homotopy equivalence then

$$f_* : H_n \mathrm{sing}_*(X) \to H_n \mathrm{sing}_*(Y)$$

is an isomorphism for all n.

> If $f, g : X \to Y$ are homotopic continuous maps, then
>
> $$f_* = g_*$$
>
> on $H_n \mathrm{sing}_*(X)$, implying
>
> $$H_n \mathrm{sing}_* : \mathsf{hTop} \to \mathsf{Ab}$$
>
> is a functor.

implies

$$(fg)_* = \mathsf{Id}_* = f_* g_*$$

Current note has 0 direct children and 0 total descendants.

- structures based on sets
    - Topological spaces

- complex sing
  - Singular chain complex of a topological space over $\mathbb{Z}$
    - Singular chain complex over $\mathbb{Z}$ functor on topological spaces
      - Homotopy invariance of singular homology functor on topological spaces

And it has 2 siblings.

- structures based on sets
  - Topological spaces
    - complex sing
      - Singular chain complex of a topological space over $\mathbb{Z}$
        - Singular chain complex over $\mathbb{Z}$ functor on topological spaces
          - Homotopy invariance of singular homology functor on topological spaces
          - Exact sequences in singular chain complex from inclusions