# Differential forms relative to a smooth map

**Definition.** Differential forms relative to a smooth map

Let $f : S \to M$ be a smooth map between smooth manifolds. Then $(\Omega^\bullet[f], \mathrm{d})$

$$\Omega^\bullet[f] := \bigoplus \Omega^k[f]$$
$$\Omega^k[f] := \Omega^k(M) \oplus \Omega^{k-1}(S)$$
$$\mathrm{d}(\omega, \theta) := (\mathrm{d}\omega, f^*\omega - \mathrm{d}\theta)$$

is the **cocomplex of differential forms relative to the map** $f$.

Should we change the definition to $\mathrm{d}\theta - f^*\omega$?

- $$\mathrm{d}^2 = 0$$
- Let $(\omega, \theta)$ be a **form** in $\Omega^k[f]$ then its derivative
$$\mathrm{d}(\omega, \theta) := (\mathrm{d}\omega, f^*\omega - \mathrm{d}\theta)$$
is a form.
- Let $(\omega, \theta)$ be a **closed form** in $\Omega^k[f]$
$$\mathrm{d}(\omega, \theta) := (\mathrm{d}\omega, f^*\omega - \mathrm{d}\theta) = 0$$
which says, $\omega$ is a closed form on $M$ which when pulled back to $S$ becomes an exact form $\mathrm{d}\theta$ on $S$.
  - Take any closed form $\omega$ on $M$ which becomes $\mathrm{d}\theta$ when pulled back to $S$, then that implies $\mathrm{d}(\omega, \theta) = 0$.
- we have a cohomology of the cocomplex $H^*(\Omega^*[f])$
- Let $(\alpha, \beta) := \mathrm{d}(\omega, \theta) := (\mathrm{d}\omega, f^*\omega - \mathrm{d}\theta)$ be an **exact form** in $\Omega^{k+1}[f]$, then $\alpha = \mathrm{d}\omega$ is an exact form on $M$ and
$$\beta = f^*\omega - \mathrm{d}\theta \implies \beta - f^*\omega = -\mathrm{d}\theta$$
so $\beta - f^*\omega$ is an exact form on $S$. If we pull $\alpha$,
$$f^*\alpha = f^*\mathrm{d}\omega = \mathrm{d}f^*\omega = \mathrm{d}\beta$$
Hence, $\alpha$ is an exact form on $M$ which pulled onto $S$ becomes $\mathrm{d}\beta$.
  - Let $\alpha$ is an exact form on $M$ which pulled onto $S$ becomes $\mathrm{d}\beta$ which means
  $$\alpha = \mathrm{d}\omega, f^*\alpha = f^*\mathrm{d}\omega = \mathrm{d}f^*\omega = \mathrm{d}\beta$$
  then
  $$\mathrm{d}(\omega, ?) = (\alpha, \beta)$$

| **On** $\Omega^k(M)$ | **On** $\Omega^k[f]$ |
|---|---|
| $\omega$ is a closed form on $M$ which when pulled back to $S$ becomes an exact form $$\mathrm{d}\theta = f^*\omega$$ on $S$ | closed form $(\omega, \theta)$ that is $$\mathrm{d}(\omega, \theta) := (\mathrm{d}\omega, f^*\omega - \mathrm{d}\theta) = 0$$ |
| $\alpha = \mathrm{d}\omega$ is an exact form on $M$ which pulled onto $S$ becomes $\mathrm{d}\beta$ | $(\alpha, \beta) = (\mathrm{d}\omega, \beta)$ whose derivative is $(0, f^*\mathrm{d}\omega - \mathrm{d}\beta) = (0, \mathrm{d}(f^*\omega - \beta))$ |
| $\alpha = \mathrm{d}\omega$ is an exact form on $M$ such that $f^*\omega - \beta = \mathrm{d}\theta$ is exact on $S$ | $$(\alpha, \beta) = \mathrm{d}(\omega, \theta) := (\mathrm{d}\omega, f^*\omega - \mathrm{d}\theta)$$ is an exact form |
| | $[(\omega, \theta)]$ |

- we have the short exact sequence in cocomplex
$$\begin{array}{ccccccccc} 0 & \to & \Omega^{k-1}(S) & \xrightarrow{\alpha} & \Omega^k[f] & \to & \Omega^k(M) & \to & 0 \\ & & \theta & \mapsto & (0, \theta) & & & & \\ & & & & (\omega, \theta) & \mapsto & \omega & & \end{array}$$
which gives a long exact sequence in cohomology

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- Chain complex of differential forms relative to a submanifold

Current note has 0 direct children and 0 total descendants.

- structures based on sets
    - manifolds
        - smooth $\mathbb{R}$-manifold
            - Tangents on a smooth manifold
                - Differential forms on smooth manifolds
                    - Three different relative chain complexes of differential forms
                        - Differential forms relative to a smooth map

And it has 2 siblings.

- structures based on sets
    - manifolds
        - smooth $\mathbb{R}$-manifold
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                    - Three different relative chain complexes of differential forms
                        - Differential forms relative to a smooth map
                        - Chain complex of differential forms relative to a submanifold