Info

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# $PSL(2)(\mathbb{Z})$

## $PSL(2)(\mathbb{Z}) \curvearrowright \mathbb{R}\boldsymbol{H}^2$

$$\begin{tikzcd}
SL(2,\mathbb{R}) \arrow[r] \arrow[d] & \mathbb{R}H^2 \arrow[d] \\
SL(2,\mathbb{Z})\backslash^{SL(2,\mathbb{R})} \arrow[r] & SL(2,\mathbb{Z})\backslash^{\mathbb{R}H^2}
\end{tikzcd}$$

The action is not free!

https://roywilliams.github.io/play/js/sl2z/

```
FareySymbol(Gamma(1)).fundamental_domain()
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> Find $\gamma_0 \in SL(2)(\mathbb{Z})$ such that
>
> $$\Im(\gamma(z)) \leq \Im(\gamma_0(z))$$
>
> for all $\gamma \in SL(2)(\mathbb{Z})$.

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We have

$$PSL(2)(\mathbb{Z}) = \left\langle z \mapsto z+1, -\frac{1}{z} \right\rangle$$

and therefore

$$SL(2)(\mathbb{Z}) = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\rangle$$

Let $H = \langle g, h \rangle$.

- > Find $\gamma_0 \in H$ such that $\Im(\gamma(z)) \leq \Im(\gamma_0(z))$ for all $\gamma \in H$.
- > This implies $HR = H^2_{\mathbb{U}}$
- Let $\gamma \in SL(2)(\mathbb{Z})$. Then there exists $\gamma' \in H$ such that
  $$\gamma'\gamma(i) \in R$$
- Then
  $$\gamma'\gamma(i) = 2i \pm b$$
  so $\gamma'\gamma \in \{I, -I\}$
- Thus $\gamma \in H$.

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- Let $z \in \operatorname{int}(R)$ and $\gamma \in \Gamma$ such that $\gamma(z) \in R$.
  - For $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $c \neq 0$.
  - As before
    $$\frac{\sqrt{3}}{2} \leq \Im(\gamma(z)) = \frac{\Im(z)}{|cz+d|^2} = \frac{y}{|c(x+iy)+d|^2}$$
    implying
    $$|c| \leq \frac{\sqrt{3}}{2} \implies c \in \{-1, 1\}$$
  -

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$$z \mapsto z+1 \mapsto -\frac{1}{z+1} \mapsto z+1$$

## $SL(2)(\mathbb{Z})\backslash^{\mathbb{R}H^2}$

[1]

## $PSL(2)(\mathbb{Z}) \curvearrowright \boldsymbol{P}^1$

We have

$$PSL(2)(\mathbb{Z}) \curvearrowright \mathbb{Q}\boldsymbol{P}^1 = \mathbb{Q} \cup \{\infty\}$$

and

$$PSL(2)(\mathbb{Z}) \curvearrowright \mathbb{R}\boldsymbol{P}^1 = \mathbb{R} \cup \{\infty\}$$

Therefore, $PSL(2)(\mathbb{Z})$ acts on the set $\mathbb{R} \setminus \mathbb{Q}$ of all irrational numbers.

Consider the positive and negative irrational numbers

$$(-\infty, 0) \setminus \mathbb{Q}$$

and

$$(0, \infty) \setminus \mathbb{Q}$$

- Under

  $$z \mapsto -\frac{1}{z}$$

  we have $(-\infty, 0) \setminus \mathbb{Q} \mapsto (0, \infty) \setminus \mathbb{Q}$
- Under

  $$z \mapsto -\frac{1}{z+1}$$

  we have $(-\infty, 0) \setminus \mathbb{Q} \mapsto (0, \infty) \setminus \mathbb{Q}$

Therefore

$$\begin{aligned} PSL(2)(\mathbb{Z}) &= \left\langle -\frac{1}{z} \right\rangle * \left\langle -\frac{1}{z+1} \right\rangle \\ &\cong \mathbb{Z}_2 * \mathbb{Z}_3 \end{aligned}$$

## $PSL(2)(\mathbb{Z})$ is maximal discrete

[2]

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Current note has 0 direct children and 0 total descendants.

- squishy
  - $SL$
    - $PSL(2)(\mathbb{Z})$

And it has 15 siblings.

- squishy
  - $SL$
    - SL(2)
    - $SL(2)(\mathbb{C})$
    - $PSL(n, \mathbb{C})$
    - $SL(2)(\mathbb{R})$
    - $SL(2)(\mathbb{R}) / SL(2)(\mathbb{Z})$
    - $H_q$
    - $PSL(2)(\mathbb{Z})[2]$
    - $[PSL(2)(\mathbb{Z})[2], PSL(2)(\mathbb{Z})[2$]]
    - $PSL(2)(\mathbb{Z})[N]$
    - $SL(3, \mathbb{R})$
    - $SL(2)(\mathbb{Z})$
    - $PSL(2)(\mathbb{Z})$
    - Congruent subgroups of $PSL(2)(\mathbb{Z})$
    - $SL(n, \mathbb{Z})$
    - $SL(2)(\mathbb{Z}_2)$

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1. https://math.stackexchange.com/questions/3087566/what-is-t1-mathbb-h2-psl-2-mathbb-z ↩
2. lie groups - Is $SL_2(\mathbb{Z})$ a maximal discrete subgroup in $SL_2(\mathbb{R})$? - Mathematics Stack Exchange ↩