> Info
>
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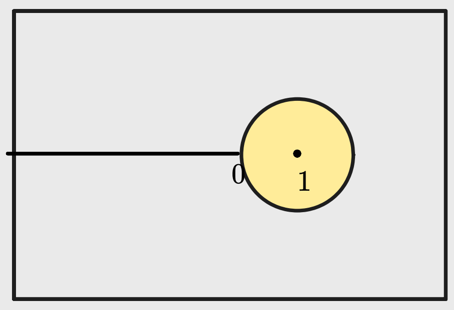
Weierstrass' primary factors

Definition. Weirstrass' primary factors

$$\begin{aligned} E_0(z) &:= 1 - z \\ E_n(z) &:= (1-z)\exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^n}{n}\right) \end{aligned}$$

for $n \in \mathbb{Z}_{>0}$

- For each $m \in \mathbb{N}$, E_n is **entire**, and only have zero at $z = 1$ of order 1.
- On unit disk $D \ni z$, as
  - 

    $$\log(1-z) = \sum_{n\geq 0} \frac{z^{n+1}}{n+1} \quad \text{on } |z| < 1$$

    - $$\log(z) = \log(1-(1-z)) = \sum_{n\geq 0} \frac{(1-z)^{n+1}}{n+1} \quad \text{on } |1-z| < 1$$

  we have

  $$E_m(z) \xrightarrow{m\to\infty} (1-z)\exp\left(\log(1-z)\right) = 1$$

  and indeed

  $$E_m \xrightarrow{\text{uniformly as } m\to\infty} 1 \text{ on } D$$

bound on $1 - E_n$

> **Proposition:** The functions
>
> > **Definition.** Weirstrass' primary factors
> >
> > $$\begin{aligned} E_0(z) &:= 1 - z \\ E_n(z) &:= (1-z)\exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^n}{n}\right) \end{aligned}$$
> >
> > for $n \in \mathbb{Z}_{>0}$
>
> satisfy
>
> $$|1 - E_n(z)| \leq |z|^{n+1}$$
>
> on the unit disk D for all n.

For $m = 0$

$$|1 - (1 - z)| = |z|^1$$

For $m \geq 1$ we have

$$E_m'(z) = -z^m \exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^m}{m}\right)$$

which means

$$(1 - E_m)'(z) = z^m \exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^m}{m}\right)$$

.

- As

  $$\begin{aligned} \frac{1 - E_m(z)}{z^{m+1}} &= \sum_{n\geq 0} a_n z^n \\ 1 - E_m(z) &= \sum_{n\geq 0} a_n z^{n+m+1} \end{aligned}$$

  we have

  $$\begin{aligned} \sum_{n\geq 0} \frac{a_n}{n+m+1} z^{n+m} &= -z^m \exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^m}{m}\right) \\ &= -z^m \sum_{n\geq 0} \frac{1}{n!}\left(z + \frac{z^2}{2} + \cdots + \frac{z^m}{m}\right)^n \end{aligned}$$

  Thus

  $$a_n \geq 0$$

- So

  $$(1 - E_m)(0) = 0$$

and $1 - E_m$ has a zero of order $m+1$ at $z = 0$. Thus

$$\frac{1 - E_m(z)}{z^{m+1}}$$

has a removable singularity on 0, so it extends to an entire function

$$\begin{aligned} \frac{1 - E_m(z)}{z^{m+1}} &= \sum_{n \geq 0} a_n z^n \\ \left| \frac{1 - E_m(z)}{z^{m+1}} \right| &\leq \sum_{n \geq 0} |a_n||z|^n \leq \sum_{n \geq 0} |a_n| \text{ on } D \\ &= \sum_{n \geq 0} a_n \\ &= \frac{1 - E_m(1)}{1^{m+1}} = 1 \end{aligned}$$

- Therefore,

$$|z| \leq 1 \implies |1 - E_m(z)|M = |z|^{m+1}$$

Current note has 0 direct children and 0 total descendants.

- squishy
  - pow
    - Weierstrass' primary factors

And it has 28 siblings.

- squishy
  - pow
    - Angle function
    - Bessel functions
    - $$\cos\sqrt{z}$$
    - The complex logarithm
    - $\sum_{n \geq 0} z^{n!}$
    - $z^3 - 3z$
    - $$\frac{r_0}{1 + \epsilon \cos b\theta}$$
    - $\cos\left(\pi \frac{p}{q}\right)$
    - $\frac{\cos x^k}{x}$
    - Elliptic integrals and Jacobi elliptic functions
    - The complex exponential $\exp(z)$
    - $\exp\left(\frac{z+1}{z-1}\right)$
    - e^{-x^2}
    - [Only sum terms that are multiples of N in $\mathbb{C}[[z](https://rupadarshiray.github.io/notes/YchnxQ9SF1RsuKjBntMv8a.$]]
    - $\int \frac{1}{x^2+bx+c}$
    - $\frac{\exp(z)}{1-z}$
    - $\sin x$
    - $\sin(x \sin(x))$
    - $\frac{\sin x}{x}$
    - stock.sin x by xk
    - $\sin x^2$
    - $\sqrt{z}$
    - $\sqrt{z^2 - 1}$
    - Theta functions
    - Weierstrass elliptic function $\wp$
    - Weierstrass' primary factors
    - $x^2 \sin \frac{1}{x}$
    - $x + x^2 \sin \frac{1}{x}$