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$\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ of functions $f, g : U \subseteq \mathbb{R} \rightarrow \mathbb{R}$

For a differentiable function $f : (-a, a) \rightarrow \mathbb{R}$ we know

$$f(0) = 0 \implies \lim_{x \rightarrow 0} \frac{f(x)}{x} = f'(0)$$

by definition.

(L'Hopital) Let $f, g : (a, b) \rightarrow \mathbb{R}$ be differentiable, $g, g' \neq 0$ and $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = A$. If

$$\begin{cases} \lim_{x \rightarrow a} f(x) = 0 = \lim_{x \rightarrow a} g(x) \\ \text{or} \\ \lim_{x \rightarrow a} g(x) = \infty \end{cases} \implies \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = A$$

Let $f, g : (a, b) \rightarrow \mathbb{R}$ be differentiable, $g, g' \neq 0$.

- Let $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = A$
 - we have $\forall \epsilon > 0, \exists c_\epsilon$ such that

$$\theta \in (a, c_\epsilon) \implies \left| \frac{f'(\theta)}{g'(\theta)} - A \right| < \frac{\epsilon}{2}$$

- For any $(x, y) \subset (a, c_\epsilon)$, then by Ratio MVT, $\exists \theta \in (x, y)$ such that

$$\begin{aligned} \frac{f'(\theta)}{g'(\theta)} &= \frac{f(x) - f(y)}{g(x) - g(y)} \\ \implies \left| \frac{f'(\theta)}{g'(\theta)} - A \right| &= \left| \frac{f(x) - f(y)}{g(x) - g(y)} - A \right| < \frac{\epsilon}{2} \end{aligned}$$

- Hence, $\forall \epsilon, \exists c_\epsilon$ such that

$$(x, y) \subset (a, c_\epsilon) \implies \left| \frac{f(x) - f(y)}{g(x) - g(y)} - A \right| < \frac{\epsilon}{2}$$

- If $\lim_{x \rightarrow a} f(x) = 0 = \lim_{x \rightarrow a} g(x)$, we have
 - $\forall \epsilon \exists \delta_\epsilon^1$

$$x \in (a, a + \delta_\epsilon^1) \implies \left| \left(\frac{f(x) - f(y)}{g(x) - g(y)} - A \right) - \left(\frac{f(y)}{g(y)} - A \right) \right| < \frac{\epsilon}{2}$$

- $\forall \epsilon$ let $\delta_\epsilon := \min(\delta_\epsilon^1, c_\epsilon)$,

$$|y - a| < \delta_\epsilon$$

- for any $x < y$ we have $(x, y) \subset (a, c_\epsilon)$ and $|x - a| < \delta_\epsilon < \delta_\epsilon^1$ we have both the previous inequalities so

$$\left| \frac{f(y)}{g(y)} - A \right| < \epsilon$$

Current note has 0 direct children and 0 total descendants.

- [stem](#)
 - [subobjects of and functions on \$\mathbb{R}^n, S^n, \mathbb{C}^n\$ and its quotients \$\Gamma \backslash \mathbb{R}^n, \Gamma \backslash \mathbb{C}^n\$](#)
 - [R1](#)
 - $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ of functions $f, g : U \subseteq \mathbb{R} \rightarrow \mathbb{R}$

And it has 5 siblings.

- [stem](#)
 - [subobjects of and functions on \$\mathbb{R}^n, S^n, \mathbb{C}^n\$ and its quotients \$\Gamma \backslash \mathbb{R}^n, \Gamma \backslash \mathbb{C}^n\$](#)
 - [R1](#)
 - [Types of discontinuities of functions on \$\mathbb{R}\$](#)
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 - [Fixed points of functions \$U \subseteq \mathbb{R} \rightarrow U\$](#)
 - $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ of functions $f, g : U \subseteq \mathbb{R} \rightarrow \mathbb{R}$
 - [Total variation and sequence of continuous functions](#)