Info

This note, that was found here, as a part of a collection
is written (completely with human hands) by Rupadarshi Ray,
was created on November 5, 2023 4:18:58 PM,
was last modified on May 24, 2026 6:43:22 PM, and
was exported on September 7, 2026 3:31:18 PM.

Integrable Hamiltonian flows

Definition. Integrable Hamiltonian flows

Let (M, ω) be a symplectic manifold and

$$f_1, \dots, f_k \in \mathcal{C}^\infty(M)$$

such that

- *(independent)* $\mathrm{d}f_1, \dots, \mathrm{d}f_k$ are linearly independent on T_p^*M for all $p \in M$
- *(pairwise in involution)* $\{f_i, f_j\} = 0$ for $i, j \leq \dim M/2$

then

if	the flow of $f_1, \dots, f_{\max\{\dim M/2, k\}}$ is called
$k < \dim M/2$	(partially) k-integrable
$k = \dim M/2$	completely integrable
$k > \dim M/2$	super k-integrable
$k = \dim M$	maximally super-integrable

Therefore, a **completely integrable** Hamiltonian flow is a Lie algebra homomorphism

$$\begin{aligned}(\mathbb{R}^{\dim M/2}, 0) &\to (\mathcal{C}^\infty(M), \{\ ,\ \}) \\ e_i &\mapsto f_i\end{aligned}$$

such that $f_1, \dots, f_{\dim M/2}$ are *independent* on M.

completely integrable systems

Let

$$(M, \omega, f_1, \dots, f_{\dim M/2})$$

be an *symplectic integrable system*, and $c \in \mathbb{R}^n$ be a *regular value* of

$$f := (f_1, \dots, f_{\dim M/2}) : M \to \mathbb{R}^n$$

then $f^{-1}(c)$ is a **Lagrangian submanifold** of M.
Now, for a connected component N of such an submanifold $f^{-1}(c)$ where **flows of X_{f_i} are complete**, then it is a homogeneous space for $\mathbb{R}^n$ and has affine coordinates for some $0 \leq k \leq \dim M$

$$\varphi : N \cong_{\text{Man}} \mathbb{R}^{\dim M - k} \times \mathbb{T}^k$$

in which the **vector fields X_{f_i} are linear**.
Also there exists a a open neighborhood U_N of N we have a **symplectomorphism**

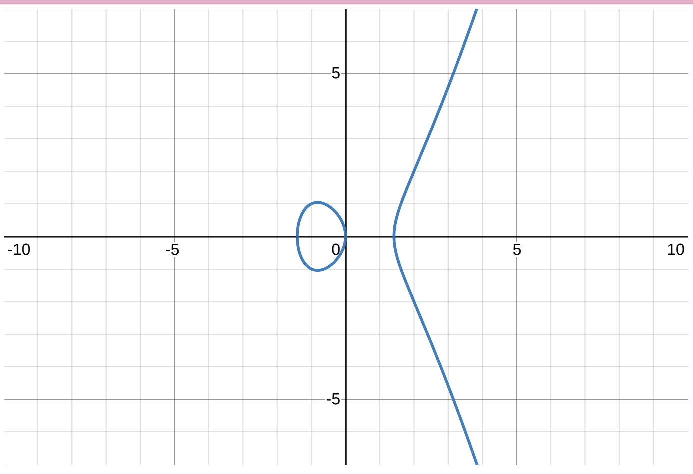
$$I \times \varphi : (U_N, \omega) \to \left(\underbrace{\mathbb{R}^{\dim M}}_{x_i} \times \mathbb{R}^{\dim M - k} \times \mathbb{T}^k, \sum \mathrm{d}x_i \wedge \mathrm{d}y_i\right)$$

such that I are constants of flow.

$f^{-1}(c)$

- $f^{-1}(c)$ is a submanifold of M, possibly not connected
- $f^{-1}(c)$ is a **Lagrangian submanifold** of M because it is $\dim M/2$ dimensional and its tangent bundle is *isotropic*.

$f^{-1}(0)$ with two connected components, where $f(x, y) = y^2 - x^3 + 2x$



- If all the vector fields X_i are **complete** on M, this is a priori a *Hamiltonian $\mathbb{R}^n$-action* with the vector fields X_i and moment map f.

on a connected component N_c of $f^{-1}(c)$

Choose a connected component N_c of $f^{-1}(c)$.

- N_c a Lagrangian submanifold of M. It might be compact, or non-compact.
- If all the vector fields X_i are **complete** on N_c, then because $[X_i, X_j] = 0$ by flow coordinates, N_c is diffeomorphic to $\mathbb{R}^l \times \mathbb{T}^m$ and the vector fields X_i look like

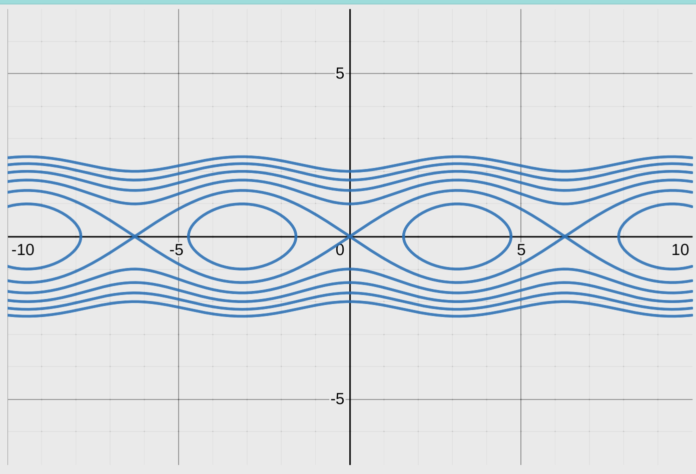
$$X_i = \omega_i \frac{\partial}{\partial \theta_i}$$

in these coordinates.

Example

$f^{-1}(c)$ for

$$\mathbb{R}^2 \to \mathbb{R}$$
$$f(x,y) = y^2 + \cos(x)$$



with N_c compact and non-compact for different values of c

when N_c is compact

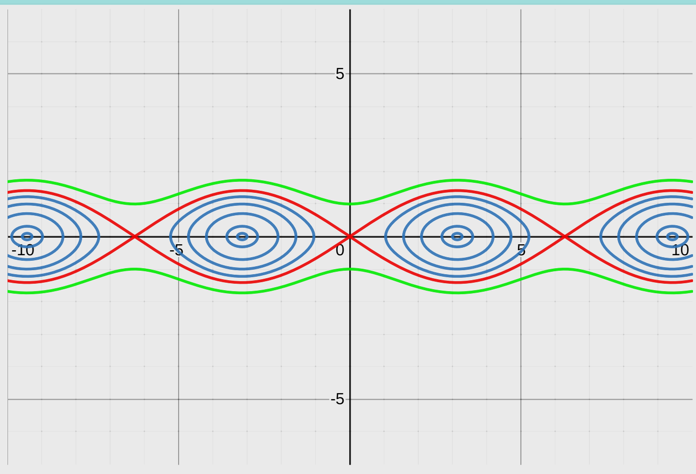
- If N_c is **compact**, X_i are **complete** on N_c and the flow coordinates of X_i defines a diffeomorphism onto the n-torus

$$\theta : N_c \cong \mathbb{T}^n$$

  where in these coordinates the vector fields X_i define a $\mathbb{T}^n$-action on N_c. This torus N_c called a **Liouville tori**.
- To ensure that we only need one f_i to be proper.

Example

$f^{-1}(c)$ for $f(x,y) = y^2 + \cos(x)$ For $c \in (-1,1)$ has a fibration of S^1 *(blue)* and for $c > 1$ fibrations by $\mathbb{R}$ *(green)*. This is *separated* by $f^{-1}(1)$ *(red)*.



- Now, we take the smooth map

$$(f,\theta) : M \to \mathbb{R}^n \times \mathbb{T}^n$$

  which satisfies

$$\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n \wedge \mathrm{d}\theta_1 \wedge \cdots \wedge \mathrm{d}\theta_n \neq 0 \text{ on } N_c$$

  - So by (generalized) inverse function theorem, there is a nbd U_c of N_c such that

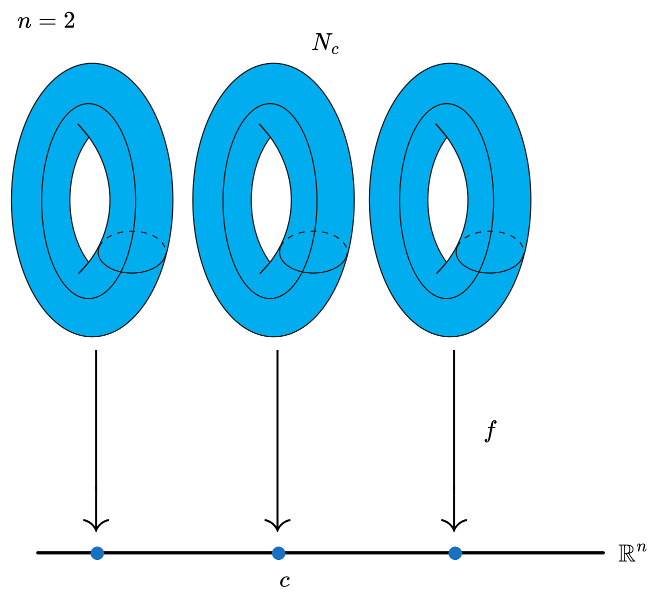
$$(f,\theta) : U_c \to \mathbb{R}^n \times \mathbb{T}^n$$

    is a diffeomorphism onto a nbd $V \subseteq \{c\} \times \mathbb{T}^n$.
  - The smooth map

$$(f,\theta) : (U_c, \omega) \to \left(\underbrace{\mathbb{R}^n}_{x_i} \times \underbrace{\mathbb{T}^n}_{y_i}, \sum \mathrm{d}x_i \wedge \mathrm{d}y_i \right)$$

    is a symplectomorphism onto its image.

- > Claim: If $f : M \to B \subseteq \mathbb{R}^n$ only has regular fibres, then there is a map
  >
  > $$(f,\theta) : f^{-1}(B) \to B \times \mathbb{T}^n$$
  >
  > which is a symplectomorphism.

- > Claim: This defines a *fibration* with fibers being Lagranagian submanifolds diffeomorphic to torus *(Liouville tori)*
  >
  > 
  >

- The vector fields look like

$$X_j = \sum_i \omega_{ij} \frac{\partial}{\partial \theta_i} \text{ on } U_c$$

- The vector fields

$$v_i \mapsto \frac{\partial}{\partial \theta_i}$$

  defines a Hamiltonian $\mathbb{T}^n$-action on U_c with moment map

$$f : (U_c, \omega) \to \mathbb{R}^n$$

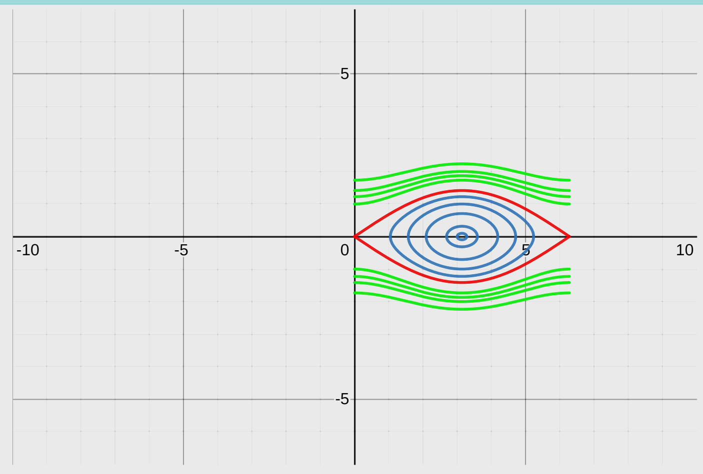
  which is smoothly $\mathbb{T}^n$-conjugate to n harmonic oscillators on the image of (f, θ) in $\mathbb{R}^n \times \mathbb{T}^n$.
    - Is this unqiue?
    - In this $\mathbb{T}^n$-action, X_j becomes the generato $\sum \omega_i v_i$.
- Hence, complete integrability (with necessary conditions) is a *local torus action* along with *Lagarangian torus fibrations*.

Example

$f^{-1}(c)$ for

$$\mathbb{R}^2/(0, 2\pi)\mathbb{Z} \to \mathbb{R}$$
$$f(x, y) = y^2 + \cos(x)$$

two fibrations for $c \in (-1, 1)$ and $c > 1$ are have S^1-fibers separated by $f^{-1}(1)$.



when N_c is not compact

- If N_c is (possibly) not compact, but still X_i are complete we take the flow coordinates

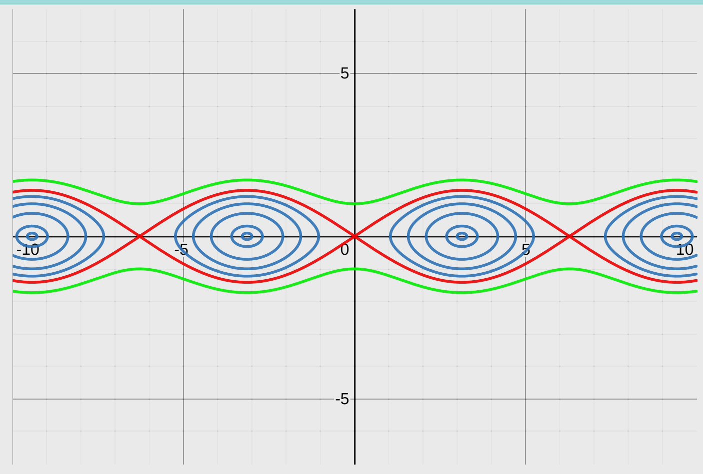
$$\theta : N_c \cong \mathbb{R}^k \times \mathbb{T}^{n-k}$$

    - The smooth map

$$(f, \theta) : M \to \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{T}^{n-k}$$

Example

$f^{-1}(c)$ for $f(x, y) = y^2 + \cos(x)$ For $c \in (-1, 1)$ has a fibration of S^1 *(blue)* and for $c > 1$ fibrations by $\mathbb{R}$ *(green)*. This is *separated* by $f^{-1}(1)$ *(red)*.



N_c is compact and non-compact for $c < 1$ and $c > 1$ and it is separated by $f^{-1}(1)$.

[1]

an integrable system is a *local* torus action?

[2]

[3]

[4]

[5]

what does this say for Hamiltonian $\mathbb{R}^n$-actions?

> - If all the vector fields X_i are **complete** on M, this is a priori a *Hamiltonian* $\mathbb{R}^n$*-action* with the vector fields X_i and moment map f.

Then for regular value c of

$$f : M \to \mathbb{R}^n$$

we have a $\mathbb{T}^n$-action on a nbd of $f^{-1}(c)$ which is conjugate to n harmonic oscillators.

(global) Hamiltonian torus-actions

Current note has 0 direct children and 0 total descendants.

- structures based on sets
    - manifolds
        - $\mathbb{R}$-manifold symplectic
            - Hamiltonian vector fields on a symplectic manifold

- Integrable Hamiltonian flows

And it has 5 siblings.

- structures based on sets
  - manifolds
    - $\mathbb{R}$-manifold symplectic
      - Hamiltonian vector fields on a symplectic manifold
        - Integrable Hamiltonian flows
        - Singular integrable flows
        - Hamiltonian Lie group action on a symplectic manifold
        - Quantization of Hamiltonian Lie group action on a symplectic manifold
        - Probability states and flows of an Hamiltonian system

---

1. Ana Cannas da Silva - Lectures on Symplectic Geometry -Springer (2009).pdf > page=123 ↩
2. Lesson 5-Toric actions, action-angle coordinates and Integrable systems on b^m-symplectic manifolds. - YouTube ↩
3. integrableandgroup..pdf (upc.edu) ↩
4. TorusSurvey2003.pdf (univ-toulouse.fr) ↩
5. Integrable systems and group actions, a love story (unige.ch) ↩