Info

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Borel measures on a smooth manifold

Definition. Borel measure on a smooth manifold

A **Borel measure** on a smooth manifold M is a *linear* form

$$\mu : \mathcal{C}_c^\infty(M) \to \mathbb{R}$$
$$f \mapsto \int_M f \, \delta\mu$$

such that it is *continuous*: if $\{f_k \mid k \in \mathbb{Z}_{>0}\} \subseteq \mathcal{C}_c^\infty(M)$ with $\|f_k\|_\infty \subseteq K(\mathrm{cpt}) \subseteq M$ then

$$\|f_k\|_\infty \xrightarrow{k\to\infty} 0 \implies \int_M f_k \, \delta\mu \xrightarrow{k\to\infty} 0$$

[1]

Definition. Lie derivative of smooth Borel measures

If μ is a Borel measure with the property that for a family of vector fields

$$X_1, \dots, X_k$$

the linear form

$$f \mapsto \mu(X_1 \dots X_k f)$$

on $\mathcal{C}_c^\infty(M)$ is a Borel measure, then μ is said to be a **smooth Borel measure**.
The space of all *smooth Borel measures* on M

$$\mathcal{B}M$$

is a $\mathcal{C}^\infty(M)$-module by

$$h \in \mathcal{C}^\infty(M) \implies h\mu := f \mapsto \mu(hf)$$

Definition. Lie derivative of smooth Borel measures

The **Lie derivative** of a *smooth Borel measure* μ with respect to the vector field X is the smooth Borel measure

$$\mathcal{L}_X \mu := f \mapsto -\mu(Xf)$$

Thus

$$\mathcal{L}_X : \mathcal{B}M \to \mathcal{B}M$$

and

$$\mathcal{L} : \mathrm{Vec}(M) \to \mathrm{End}(\mathcal{B}M)$$

We consider $\mathcal{C}^\infty(M)$-linear maps

$$\Omega^{\dim M}(M) \to \mathcal{B}M$$

that are **flat**, that is, commutes with Lie derivative with respect to all smooth vector fields.

Definition. To any open set $U \subseteq M$ the assignment

$$U \mapsto \mathcal{B}(U)$$

defines a sheaf of **smooth Borel measures** denoted by $\mathcal{B}_M$

Definition. Definition

To any open set $U \subseteq M$ we assign the set of all *flat* homomorphisms

$$\Omega_M^{\dim M}(U) \to \mathcal{B}_M(U)$$

is a sheaf of $\mathbb{R}$-vector spaces denoted by

$$OZ_M$$

> The image of $\Omega_M^{\dim M}$ in sheaf $\mathcal{B}_M$ by any non-zero flat homomorphism is called the **sheaf of densities** $\mathcal{D}_M$ and thus
>
> $$\Omega_M^{\dim M} \otimes OZ_M \cong \mathcal{D}_M$$

Current note has 0 direct children and 0 total descendants.

- structures based on sets
  - manifolds
    - smooth $\mathbb{R}$-manifold
      - Borel measures on a smooth manifold

And it has 19 siblings.

- structures based on sets
  - manifolds
    - smooth $\mathbb{R}$-manifold

- Smooth Lie group action on a smooth manifold
- Borel measures on a smooth manifold
- G-bundles over smooth manifolds $G \curvearrowright P \to X$
- Vector bundle on smooth manifolds
- Curves in smooth manifolds
- Density on smooth manifolds
- Functions between smooth manifolds
- Functions from smooth manifolds to $\mathbb{R}$
- Flows on a smooth manifold
- Locally $G \curvearrowright X$-manifolds
- Smooth singular homology of a smooth manifold
- Isomorphisms and automorphisms of smooth manifolds
- Lagrangian flows
- Local transformation rules
- Moduli space of Riemannian metrics on a smooth manifold
- Moduli space of quotients
- Quotient manifolds
- Submanifolds of smooth manifolds
- Tangents on a smooth manifold

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1. S Ramanan - Global Calculus-American Mathematical Society (2005).pdf > page=86 ↩