> Info
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Bounding holomorphic functions linearly

> **(Swartz inequality: bounded holomorphic function on disk is linearly bounded)**
> Let $f : D \to D$ be holomorphic that fixes 0, then $|f(z)|$ is atmost $|z|$, that is,
>
> $$f \in \mathscr{O}(D, D), f(0) = 0 \implies |f(z)| \leq |z|$$
>
> Moreover, f is a rotation $f \equiv cz, |c| = 1 \iff$ there exists $z \in D$ such that $|f(z)| = |z|$.

> *Steady-state thermal* interpretation of Swartz inequality
>
> As
>
> $$z \mapsto \log |f(z)|$$
>
> is a steady-state tempature field with a freezor at $z = 0$ producing negative temparature on the unit disk
>
> $$\log |f(z)| \leq 0 \qquad z \in D$$
>
> Removing the freezor/add a heater at $z = 0$
>
> $$z \mapsto \log |f(z)| - \log |z| = \log \left| \frac{f(z)}{z} \right|$$
>
> By *removable singularity*, we know $\log \left| \frac{f(z)}{z} \right| \geq -M$ for some lower bound.
> Because
>
> $$\begin{aligned} &\forall z \in \partial D, |z| = 1 \\ &\implies \log |z| = 0 \\ &\implies \log \left| \frac{f(z)}{z} \right| = \log |f(z)| \leq 0 \end{aligned}$$
>
> By
>
> > *Steady-state thermal* interpretation of Maximum of holomorphic functions on is never attained on the interior
> >
> > As
> >
> > $$z \mapsto \log |f(z)|$$
> >
> > is a harmonic function, steady-state temperature, there cannot be any maximum on the interior of a domain.
>
> this field is also negative on the unit disk
>
> $$\log \left| \frac{f(z)}{z} \right| \leq 0$$

This produces a proof for the special case when f extends to ∂D.

- Suppose the holomorphic mapping

  $$f : D \to D$$

  extends to a neighbourhood U of $\overline{D}$.
  - Then

    $$\begin{aligned} f : \overline{D} \to \overline{f(D)} \subseteq \overline{D} \\ \implies \forall z \in \overline{D}, |f(z)| \leq 1 \end{aligned}$$

  - The function

    $$z \mapsto \frac{f(z)}{z}$$

    is holomorphic on U
  - Therefore,

    $$\forall z \in \partial D, \left| \frac{f(z)}{z} \right| = |f(z)| \leq 1$$

  - By

    > **(Maximum of holomorphic functions on is never attained on the interior)** Let $f \in \mathscr{O}(U)$ be a **non-constant** holomorphic function on a open set $U \subseteq \mathbb{C}$. Then $|f|$ has no local maximum on U.
    > For $f \in \mathscr{O} \cap \mathcal{C}(\overline{U})$ where $U \subseteq \mathbb{C}$ is a bounded open subset, $\|f\|_\infty$ is attained on ∂U.

    we conclude

    $$\forall z \in D, \left| \frac{f(z)}{z} \right| \leq 1$$

For the general case,...

- Consider a holomorphic mapping

  $$f : D \to D$$

  such that $f(0) = 0$.

- Then

$$f(z) \equiv zg(z)$$

for a holomorphic $g : D \to \mathbb{C}$.
- Applying

> **(Maximum of holomorphic functions on is never attained on the interior)** Let $f \in \mathscr{O}(U)$ be a **non-constant** holomorphic function on a open set $U \subseteq \mathbb{C}$. Then $|f|$ has no local maximum on U.
> For $f \in \mathscr{O} \cap \mathcal{C}(\overline{U})$ where $U \subseteq \mathbb{C}$ is a bounded open subset, $\|f\|_\infty$ is attained on ∂U.

we conclude for each $r \in (0,1)$

$$g\Big|_{\overline{B(r,0)}}$$

attains its maximum should lie at some point $z_r \in \partial B(r,0)$.
- We see

$$\begin{aligned}\forall r \in (0,1),\ f(z_r) \in D \\ \implies |f(z_r)| \leq 1 \\ \implies |g(z_r)| \leq \frac{1}{|z_r|} = \frac{1}{r}\end{aligned}$$

- Therefore

$$\forall r \in (0,1), \forall z \in B(r,0), |g(z)| \leq \frac{1}{r}$$

  - Thus,

$$\begin{aligned}\forall r \in (0,1), g(B(r,0)) \subseteq B\left(\frac{1}{r}, 0\right) \\ \implies g(B(1,0)) &= \bigcap_{r\in(0,1)} g(B(r,0)) \\ &\subseteq \bigcap_{r\in(0,1)} B\left(\frac{1}{r}, 0\right) \\ &= \overline{B(1,0)}\end{aligned}$$

  - *or*, for every $z \in B(1,0)$

$$\begin{aligned}\forall r > |z|, |g(z)| \leq \frac{1}{r} \\ \xrightarrow{r \to 1} |g(z)| \leq 1\end{aligned}$$

- Therefore,

$$|f(z)| \leq |z| \qquad z \in D$$

- By

> **(Open mapping theorem for open subsets in $\mathbb{C}$)** Let f be *non-constant* holomorphic function on a connected open set U then $f(U)$ is open (and connected).

  - g is non-constant $\iff$ $|g(z)| < 1$ for all $z \in D$,
  - and g is constant $\iff$ $|g(z)| = 1$ for some $z \in D$
- That is, $f(z)$ is a rotation $\iff$ $\exists z \in D$ such that

$$|f(z)| = |z|$$

> **Proposition:** Let $f : D \to D$ be holomorphic that fixes 0,
>
> $$f \in \mathscr{O}(D,D), f(0) = 0 \implies \begin{cases}|f'(0)| \leq 1 \\ |f(z)| \leq |z|\end{cases}$$
>
> Moreover, equality in *either* of these inequalities $\iff$ its a rotation $f(z) \equiv cz, |c| = 1$.

Analogously,

> **Proposition:** Consider a holomorphic function f such that
>
> $$f : B(R_0, z_0) \to B(M_0, f(z_0))$$
>
> that is
>
> $$|f(z) - f(z_0)| < M_0 \qquad |z - z_0| < R_0$$
>
> Then
>
> $$|f(z) - f(z_0)| \leq \frac{M_0}{R_0}|z - z_0| \qquad |z - z_0| < R_0$$

By

> **(Swartz inequality: bounded holomorphic function on disk is linearly bounded)** Let $f : D \to D$ be holomorphic that fixes 0, then $|f(z)|$ is atmost $|z|$, that is,
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> $$f \in \mathscr{O}(D,D), f(0) = 0 \implies |f(z)| \leq |z|$$
>
> Moreover, f is a rotation $f \equiv cz, |c| = 1 \iff$ there exists $z \in D$ such that $|f(z)| = |z|$.

on

$$g(z) := f(R_0 z) - f(z_0)$$

> **(Swartz-Pick inequality)** Any holomorphic map
>
> $$g : D \to D$$

satisfies

$$p, q \in D \implies \left| \frac{g(p) - g(q)}{1 - \overline{g(p)} g(q)} \right| \leq \left| \frac{p - q}{1 - \bar{p} q} \right|$$

and

$$z \in D \implies \frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}$$

We may rephrase it in distance geometric terms: the *Mobius distance* on D

$$d_{\mathsf{Mob}}(p, q) := \left| \frac{p - q}{1 - \bar{p} q} \right|$$

then

$$p, q \in D \implies d(g(p), g(q)) \leq d(p, q)$$

for any holomorphic $g : D \to D$.

Any holomorphic map

$$f : D \to D, \ \overline{f(D)} \subset_{\mathsf{cpt}} D$$

then it is a contraction in the hyperbolic metric on D

$$d_{\mathsf{hyp}}(f(p), f(q)) \leq d_{\mathsf{hyp}}(p, q)$$

Any *proper* holomorphic map $g : D \to D$ is of the form

$$g(z) = e^{i\theta} \prod_{i=1}^{n} \frac{z - a_i}{1 - \bar{a}_i z}$$

which is, in particular, surjective and extends to a continuous function $\bar{D} \to \bar{D}$.

As $g : D \to D$ is proper, that is, preimage of compact sets are compact, we know

$$r < 1 \implies f^{-1}(\overline{B_r(0)}) \subset B_\delta(0) \text{ for } \delta < 1$$

Current note has 0 direct children and 0 total descendants.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $_\Gamma\backslash^{\mathbb{R}^n}, _\Gamma\backslash^{\mathbb{C}^n}$
    - Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
      - mapping
        - Bounding holomorphic functions linearly

And it has 13 siblings.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $_\Gamma\backslash^{\mathbb{R}^n}, _\Gamma\backslash^{\mathbb{C}^n}$
    - Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
      - mapping
        - Bounding derivative of holomorphic mappings
        - Bounding holomorphic functions linearly
        - Branch points and ramification index of holomorphic functions
        - Disk in holomorphic image of disk
        - Size of holomorphic image of disks
        - Maximum and minimum of holomorphic functions
        - Fibers of holomorphic mappings
        - Non-constant holomorphic functions locally look like w^k and are open mappings
        - Biholomorphic mapping onto disk
        - Conformal radius of pointed simply connected open subsets of $\mathbb{C}$
        - Holomorphic functions on punchered disk
        - Holomorphic function whose singularities are not isolated
        - A holomorphic map branched over $\mathbb{Q}$