Info

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Recurrence in iterations of a measure preserving endomorphism

[1]

- Let $(X, \mathcal{A}, \mu)$ be a **probability space** and $T : X \to X$ be a μ-preserving transformation. For a measurable subset $A \subseteq X$, consider the subsets
$$A, T^{-1}(A), T^{-2}(A), \ldots$$
which are all measurable subsets of X.
  - As
$$T^{-n}(A) \subseteq X$$
we have
$$\begin{aligned} &\bigcup_{n\in\mathbb{N}} T^{-n}(A) \subseteq X \\ \implies \mu\Big(&\bigcup_{n\in\mathbb{N}} T^{-n}(A)\Big) \leq \mu(X) < \infty \end{aligned}$$
  - But as $\mu(A) = \mu(T^{-n}(A))$, if $T^{-n}(A)$ are all disjoint sets we have
$$\mu(X) \geq \sum_{n\geq 0} \mu(T^{-1}(A)) = \sum_{n\geq 0} \mu(A)$$
  - Hence we conclude
$$\mu(A) > 0 \implies \{T^{-n}(A) \mid n \in \mathbb{N}\} \text{ are not disjoint}$$
which is $\iff$ there exists $n, m \in \mathbb{N}, n > m$
$$x \in T^{-n}(A) \cap T^{-m}(A) \implies T^n(x) \in AT^{m-n}(T^n x) = T^m(x) \in A$$
that is there exists points in A return back to A.
  - The *contrapositive* of the above statement is
$$\{T^{-n}(A) \mid n \in \mathbb{N}\} \text{ are disjoint} \implies \mu(A) = 0$$
- Thus for a subset $B \subseteq X$ with $\mu(B) > 0$, there are points in $B \cap T^{-n}(B)$ for some n, that means there are points in B that return back to B after some iterations.
- Consider the set $A \subset B$ of points that do not return to B
$$A := \{x \in B \mid \forall n > 0 : T^n(x) \notin B\}$$
, so this means points of A do not return to A either. If $T^{-n}(A)$ are *not* pairwise disjoint for all $n \in \mathbb{N}$ then there exists $n, m \in \mathbb{N}, n > m$
$$x \in T^{-n}(A) \cap T^{-m}(A) \implies T^n(x) \in A, T^{m-n}(T^n x) = T^m(x) \in A$$
meaning the point $T^n(x)$ in A returns to A, which contradicts the definition of A. Thus $T^{-n}(A)$ are all disjoint.

Thus applying the same conclusion for the set of points in B that do *not* return back to B we show that it is a set of measure zero:

> **(weak form of Poincare recurrence)** Let $(X, \mathcal{A}, \mu)$ be a **probability space** and $T : X \to X$ be a μ-preserving transformation and $B \subseteq X$ such that $\mu(B) > 0$. Then the set of points in B that never returns to B has measure zero.

- Now consider the set of points in B that do not return to B *infinitely many times* that is
$$A_\infty := \{x \in B \mid \exists k \geq 1 : \forall n > k(T^n(x) \notin B)\}$$
  - If we consider the sets
$$\begin{aligned} A &:= \{x \in B \mid \forall n > 0(T^n(x) \notin B)\} \\ A_k &:= \{x \in B \mid T^k(x) \in B \text{ and } \forall n > k(T^n(x) \notin B)\} \end{aligned}$$
for $k \geq 1$,
  - We see
$$\begin{aligned} x \in T^{-k}(A_0) &\iff T^k(x) \in A \\ \iff T^k(x) \in B &\text{ and } \forall n > 0 \;\; T^n(T^k x) \notin B \end{aligned}$$
so
$$A_k = T^{-k}(A)$$
for all $k \geq 1$.
  - If $x \in A_\infty$ there must be a largest integer k such that $T^k(x) \in B$ so $x \in A_k$. Conversely, if $x \in B \cap A_k$, we know
$$T^n(x) \notin B$$
for all $n > k$ meaning $x \in A_\infty$.
  - Thus
$$A_\infty = B \cap \bigcup_{k\in\mathbb{N}\cup\{\infty\}} A_k$$

implying

$$A_\infty \subseteq \bigcup_{k \in \mathbb{N} \cup \{\infty\}} A_k$$

$$\implies \mu(A_\infty) \leq \mu\left(\bigcup_{k \in \mathbb{N} \cup \{\infty\}} A_k\right)$$

- And we have already proven $\{A_k \mid k \in \mathbb{N}\}$ are all disjoint
    - Consider the set $A \subset B$ of points that do not return to B

      $$A := \{x \in B \mid \forall n > 0 : T^n(x) \notin B\}$$

      , so this means points of A do not return to A either. If $T^{-n}(A)$ are *not* pairwise disjoint for all $n \in \mathbb{N}$ then there exists $n, m \in \mathbb{N}, n > m$

      $$x \in T^{-n}(A) \cap T^{-m}(A) \implies T^n(x) \in A, T^{m-n}(T^n x) = T^m(x) \in A$$

      meaning the point $T^n(x)$ in A returns to A, which contradicts the definition of A. Thus $T^{-n}(A)$ are all disjoint.
- Thus we can conclude by the previous argument

$$\mu(A_k) = 0 \implies \mu(A_\infty) \leq \sum_{k \in \mathbb{N} \cup \{\infty\}} \mu(A_k) = 0$$

(infinite Poincare recurrence) Let $(X, \mathcal{A}, \mu)$ be a **probability space** and $T : X \to X$ be a μ-preserving transformation and $B \subseteq X$ such that $\mu(B) > 0$. Then the set of points in B that returns to B *only finitely many times* has measure zero.

Current note has 0 direct children and 0 total descendants.

- structures based on sets
    - Measure spaces
        - Measure preserving endomorphisms
            - Recurrence in iterations of a measure preserving endomorphism

And it has 2 siblings.

- structures based on sets
    - Measure spaces
        - Measure preserving endomorphisms
            - Interpretation of measure preserving endomorphisms
            - Recurrence in iterations of a measure preserving endomorphism

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1. C3L3N.pdf ↩