Info

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$SU_{\mathbb{C}}(2)$

#wiki/Man/R/3

The elements of $SU(2)$ are complex 2x2 matrices for the form

$$\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$$

with $a, b \in \mathbb{C}, \bar{a}a + \bar{b}b = 1$. The inverse of this matrix is

$$\begin{bmatrix} \bar{a} & -b \\ \bar{b} & a \end{bmatrix}$$

The map

$$SU(2) \to S^3 \subseteq \mathbb{C}^2$$
$$\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \mapsto (a, b)$$

is a diffeomorphism and it is an isomorphism of Lie groups when composed with the identification

$$SU(2) \to S^3 \subset \mathbb{R}^4 \cong \mathbb{H}^4$$

- The identity matrix

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mapsto (1, 0)$$

  which is the north pole.

latitudes are conjugacy classes

Definition. Latitudes in $SU(2)$

For $c \in (-1, 1)$, the 2-spheres

$$q_c := \left\{ \begin{bmatrix} c + ix & b \\ -\bar{b} & c - ix \end{bmatrix} \middle| c \in (-1, 1), x \in \mathbb{R}, c^2 + x^2 + b\bar{b} = 1 \right\}$$

whose images in S^3 is of the form $(c + ix, b)$, are called **loci latitudes**.
These are the set of all matrices in $SU(2)$ with trace $= 2c =$ sum of their eigenvalues.

- Except for $I, -I$, we choose a matrix in $SU(2)$

$$\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$$

  - it has characteristic polynomial

$$\det \begin{bmatrix} X - a & -b \\ \bar{b} & X - \bar{a} \end{bmatrix} = (X - a)(X - \bar{a}) + b\bar{b} = X^2 - (a + \bar{a})X + 1$$

  - which is a real polynomial with roots

$$\frac{1}{2}\left(2\mathrm{Re}(a) \pm \sqrt{4\mathrm{Re}(a)^2 - 4}\right) = \mathrm{Re}(a) \pm \sqrt{\underbrace{\mathrm{Re}(a)^2 - 1}_{\leq 0}}$$
$$=: \lambda, \bar{\lambda}$$

  with $\lambda := \mathrm{Re}(a) + i\sqrt{1 - \mathrm{Re}(a)^2}$

- Let $P \in SU(2)$ with eigenvalues $\lambda, \bar{\lambda}$ with normalized eigenvectors

$$\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} -\bar{v} \\ \bar{u} \end{bmatrix}$$

  then

$$Q := \begin{bmatrix} u & -\bar{u} \\ v & \bar{u} \end{bmatrix}$$

  is in $SU(2)$ with

$$PQ = Q \begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix}$$

The

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whose images in S^3 is of the form $(c + ix, b)$, are called **loci latitudes**.
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are **conjugacy classes** in $SU(2)$ for each $c \in (-1, 1)$. The remaining conjugacy classes are $\{I\}$ and $\{-I\}$.

The **equator** of $SU(2)$ is the conjugacy class q_0, the set of all traceless matrices in the group.

> $A \in SU(2)$ is on the equator q_0, or $\operatorname{tr}(A) = 0$
>
> $$\iff$$
>
> $A^2 = -I$
>
> $$\iff$$
>
> eigenvalues of A are $i, -i$.

longitudes are one-parameter subgroups

> **Definition.** Longitudes in $SU(2)$
>
> Let
>
> $$l(v,\theta) := \cos\theta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sin\theta \begin{bmatrix} v_1 & v_2 \\ -\bar{v}_2 & \bar{v}_1 \end{bmatrix}$$
>
> be the parameterization of $\operatorname{span}\{(1,0), v\} \cap S^3$ which is a **longitude** in the 3-sphere.

- Elements of $l(v,\theta)$ can be written as

  $$\cos\theta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sin\theta \begin{bmatrix} ix & v_2 \\ -\bar{v}_2 & -ix \end{bmatrix}$$

  for some $x > 0$.
- The $U(1)$-scaling action

  $$\begin{aligned} &U(1) \curvearrowright SU(2) \\ &\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \overset{e^{i\theta}}{\longmapsto} \begin{bmatrix} e^{i\theta} a & e^{i\theta} b \\ -e^{-i\theta}\bar{b} & e^{-i\theta}\bar{a} \end{bmatrix} \end{aligned}$$

  is the left multiplication of the subgroup

  $$\begin{aligned} U(1) &\hookrightarrow SU(2) \\ e^{i\theta} &\mapsto \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \\ &= \cos\theta \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} + \sin\theta \begin{bmatrix} i & \\ & -i \end{bmatrix} \end{aligned}$$

  whose image in S^3 is a longitude.

> All the longitudes are conjugate subgroups of $SU(2)$, that is, conjugate to the subgroup $U(1) \hookrightarrow SU(2)$.

the Lie algebra of $SU(2)$ are trace zero anti-Hermitian matrcies

A trace zero anti-Hermitian 2x2 matrix looks like

$$\begin{bmatrix} ip & u \\ -\bar{u} & -ip \end{bmatrix}$$

because

$$\begin{bmatrix} p & u \\ -\bar{u} & -p \end{bmatrix}^{\dagger} = \begin{bmatrix} -ip & -u \\ \bar{u} & ip \end{bmatrix} = -\begin{bmatrix} p & u \\ -\bar{u} & -p \end{bmatrix}$$

Thus $\mathfrak{su}(2,\mathbb{C})$ is a 3 dimensional real vector space.

This is the Lie algebra of $SU(2)$ because the tangent space at I (north pole) must be the real span of the equator, which spans this space exactly.

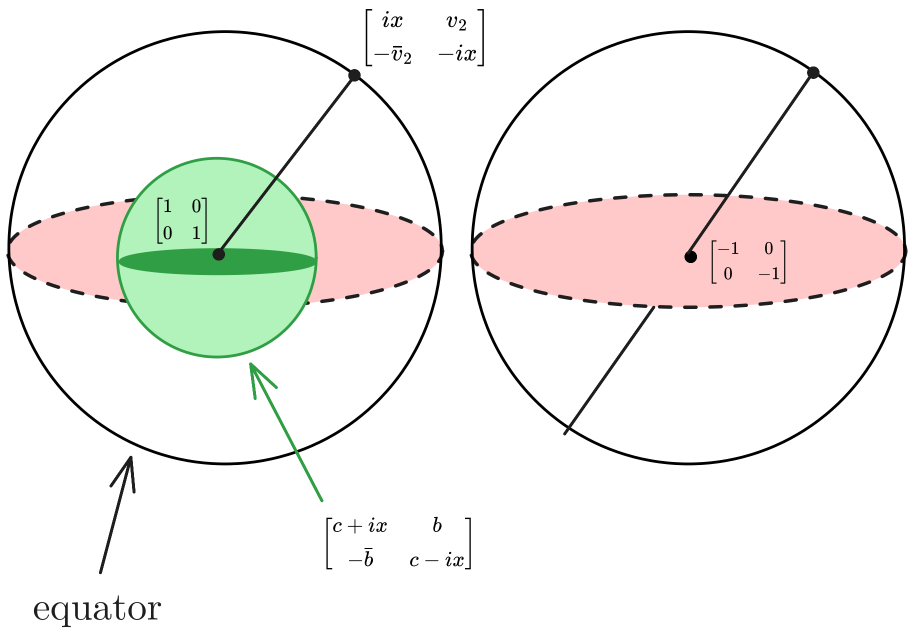
the exponential raises an element of the Lie algebra onto a longitude

By

$$\left.\frac{\mathrm{d}}{\mathrm{d}\theta}\right|_{\theta=0} \cos\theta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sin\theta \begin{bmatrix} ix & v_2 \\ -\bar{v}_2 & -ix \end{bmatrix} = \begin{bmatrix} ix & v_2 \\ -\bar{v}_2 & -ix \end{bmatrix}$$

we have the exponential

$$\exp\left(\theta \begin{bmatrix} ix & v_2 \\ -\bar{v}_2 & -ix \end{bmatrix}\right) = \cos\theta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sin\theta \begin{bmatrix} ix & v_2 \\ -\bar{v}_2 & -ix \end{bmatrix}$$



Hence we already know exponential is surjective and how much injective it is.

Through the adjoint action, $SU(2)$ acts on $\mathfrak{su}(2,\mathbb{C})$ by rotations

We define a complex inner product on $\mathfrak{su}(2,\mathbb{C})$ by

$$\langle X \mid Y \rangle := -\frac{1}{2}\operatorname{tr}(XY)$$

which gives

$$\left\langle \begin{bmatrix} ip_1 & u_1 \\ -\bar{u}_1 & -ip_1 \end{bmatrix}, \begin{bmatrix} ip_2 & u_2 \\ -\bar{u}_2 & -ip_2 \end{bmatrix} \right\rangle = -\frac{1}{2}\mathrm{tr}\begin{bmatrix} -p_1p_2 - u_1\bar{u_2} & \\ & -\bar{u}_1u_2 - p_1p_2 \end{bmatrix}$$
$$= p_1p_2 + u_1\bar{u_2}$$

The adjoint action $SU(2) \curvearrowright \mathfrak{su}(2,\mathbb{C})$ is by conjugation

$$O_U : X \mapsto UXU^{-1}$$

which is orthogonal under the inner product

$$\begin{aligned} \langle O_U X \mid O_U Y \rangle &= -\frac{1}{2}\mathrm{tr}(UXU^{-1}UYU^{-1}) \\ &= -\frac{1}{2}\mathrm{tr}(UXYU^{-1}) \\ &= -\frac{1}{2}\mathrm{tr}(XY) = \langle X \mid Y \rangle \end{aligned}$$

The map

$$\begin{aligned} \mathrm{Ad} = \mathrm{Spin} : SU(2) &\to SO(\mathfrak{su}(2,\mathbb{C}), \langle | \rangle) \cong SO(3) \\ U &\mapsto O_U \end{aligned}$$

is a surjective Lie group morphism with kernel equal to the centre of $SU(2)$ that is $\{I, -I\}$.

In the basis

$$\begin{aligned} X_1 &:= \begin{bmatrix} i & \\ & -i \end{bmatrix} \\ X_2 &:= \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \\ X_3 &:= \begin{bmatrix} & i \\ i & \end{bmatrix} \end{aligned}$$

of $\mathfrak{su}(2,\mathbb{C})$ has the commutators

$$\begin{aligned} UX_1U^{-1} &= \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}^{-1} &= \begin{bmatrix} \frac{i(\bar{a}a-\bar{b}b)}{\bar{a}a+\bar{b}b} & -\frac{2iab}{\bar{a}a+\bar{b}b} \\ -\frac{2i\bar{a}\bar{b}}{\bar{a}a+\bar{b}b} & \frac{i(-\bar{a}a+\bar{b}b)}{\bar{a}a+\bar{b}b} \end{bmatrix} \\ UX_2U^{-1} &= \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}^{-1} &= \begin{bmatrix} \frac{-\bar{a}b+\bar{b}a}{\bar{a}a+\bar{b}b} & \frac{a^2+b^2}{\bar{a}a+\bar{b}b} \\ \frac{-\bar{a}^2-\bar{b}^2}{\bar{a}a+\bar{b}b} & \frac{\bar{a}b-\bar{b}a}{\bar{a}a+\bar{b}b} \end{bmatrix} \\ UX_3U^{-1} &= \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}^{-1} &= \begin{bmatrix} \frac{i(\bar{a}b+\bar{b}a)}{\bar{a}a+\bar{b}b} & \frac{i(a^2-b^2)}{\bar{a}a+\bar{b}b} \\ \frac{i(\bar{a}^2-\bar{b}^2)}{\bar{a}a+\bar{b}b} & \frac{i(-\bar{a}b-\bar{b}a)}{\bar{a}a+\bar{b}b} \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

toroidal coordinates on S^3

$$(a, b) \in S^3 \subset \mathbb{C}^2$$

can be written in the toroidal coordinates

$$(r_1 \exp(t\theta), r_2 \exp(i\psi))$$

where $r_1^2 + r_2^2 = 1$. Thus (r_1, r_2) also live in the unit circle

$$\begin{aligned} T^3 &\to S^3 \\ (\theta, \varphi, \psi) &\mapsto (a, b) = (\cos\varphi \exp(i\theta), \sin\varphi \exp(i\psi)) \end{aligned}$$

The 3-form on $\mathbb{R}^4 = \mathbb{C}^2$

$$\begin{aligned} \lambda := &x_1 \mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_4 \\ &-x_2 \mathrm{d}x_1 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_4 \\ &+x_3 \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_4 \\ &-x_4 \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \end{aligned}$$

is invariant under $SO(4)$, so the pulled back form on S^3 is $SO(4)$ invariant as well.

$$\begin{aligned} \mathrm{d}a \wedge \mathrm{d}\bar{a} &= (\mathrm{d}x_1 + i\mathrm{d}x_2) \wedge (\mathrm{d}x_1 - i\mathrm{d}x_2) = -2i\mathrm{d}x_1 \wedge \mathrm{d}x_2 \\ &= r_1 \mathrm{d}r_1 \wedge \mathrm{d}\theta \end{aligned}$$

$$\begin{aligned} x_1 + ix_2 &= re^{i\theta} \\ \mathrm{d}x_1 + i\mathrm{d}x_2 &= e^{i\theta}\mathrm{d}r + ir\theta e^{i\theta}\mathrm{d}\theta \end{aligned}$$

$$x_1 \mathrm{d}x_2 - x_2 \mathrm{d}x_1 = (x_1^2 + x_2^2)\mathrm{d}\theta$$

So

$$\lambda = \frac{r_1^2}{-2i}\mathrm{d}\theta \wedge \mathrm{d}b \wedge \mathrm{d}\bar{b} + \frac{r_2^2}{-2i}\mathrm{d}a \wedge \mathrm{d}\bar{a} \wedge \mathrm{d}\psi$$

> **Proposition:** If $f \in L^1(SU(2))$ then
>
> $$\int_{SU(2)} f \,\mathrm{d}\mu_{\mathsf{Haar}} = \frac{1}{2\pi^2}\int_{SU(2)=S^3} f\,\lambda$$

Current note has 0 direct children and 0 total descendants.

- squishy
    - SU
        - $SU_{\mathbb{C}}(2)$

And it has 2 siblings.

- squishy
    - SU

- $SU_{\mathbb{C}}(2)$
- $SU(n, \mathbb{C})$