Info

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k-cycles of closed manifolds

Ring of closed manifolds upto cobordism

[1]

[2]

Definition. k-cycles of closed manifold

A **k-cycle of closed manifold S inside smooth manifold M** is a smooth map

$$\phi : S \to M$$

where S is a compact, oriented smooth k-manifold without boundary. Let

$$\mathcal{C}_k(M)$$

the set of all such k-cycles.

$$\mathrm{Cob}\mathcal{C}_k(M) \subset \mathcal{C}_k(M) \times \mathcal{C}_k(M)$$

such that $\phi_1 : S_1 \to M, \phi_2 : S_2 \to M \in \mathrm{Cob}\mathcal{C}_k(M)$ if there exists a cobordism : this is an equivalence relation on $\mathcal{C}_k(M)$.

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we define the set of equivalence classes

$$\mathcal{Z}_k(M) := \frac{\mathcal{C}_k(M)}{\mathrm{Cob}\mathcal{C}_k(M)}$$

with the equivalence class of $\phi : S \to M$ denoed by $[\phi : S \to M]$.

$$\mathcal{H}_k(M) := \frac{\mathcal{Z}_k(M)}{\langle [0] \in \mathcal{Z}_k(M) \rangle}$$

$$(\mathcal{Z}_k(M), \coprod)$$

is an Abelian group with $[0]$ as identity

dual forms

Definition. Poincare dual of a cycle

Let M be a oriented smooth manifold. Any k-cycle $\phi : S \to M$ gives

$$\int_S \phi^*(-) \in H^k(\mathrm{Forms}^\bullet(M), \mathrm{d})^*$$

By *Poincare duality* we identify

$$\begin{aligned} \mathrm{PD} : H^k(\mathrm{Forms}^\bullet(M), \mathrm{d})^* &\cong H^{n-k}(\mathrm{Forms}^\bullet_c(M), \mathrm{d}) \\ \int_S \phi^*(-) &\mapsto [\eta_S] \\ &: \int_M \omega \wedge \eta_S = \int_S \phi^* \omega \end{aligned}$$

Hence, we have

$$\begin{aligned} \mathrm{PD} : \mathcal{C}_k &\to H^{n-k}(\mathrm{Forms}^\bullet_c(M), \mathrm{d}) \\ (\phi : S \to M) &\mapsto [\eta_S] \end{aligned}$$

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1. Bordism.pdf (ucsd.edu) ↩
2. vitocruz.pdf (berkeley.edu) ↩