Info

This note, that was found here, as a part of a collection
is written (completely with human hands) by Rupadarshi Ray,
was created on August 2, 2025 1:00:57 AM,
was last modified on August 24, 2025 11:17:23 AM, and
was exported on September 7, 2026 3:24:49 PM.

reconstructing compact Hausdorff X from $\mathcal{C}(X,\mathbb{R})$

Let X be a compact Hausdorff space, $\mathcal{C}(X,\mathbb{R})$ be the ring of all continuous functions $X\to\mathbb{R}$ and

$$\operatorname{maxSpec}\mathcal{C}(X,\mathbb{R})$$

denote the maximal ideals in spectrum of $\mathcal{C}(X,\mathbb{R})$.

We have the surjective ring map

$$\mathrm{ev}_x:\mathcal{C}(X,\mathbb{R})\to\mathbb{R}$$

whose kernel is a maximal ideal $\mathfrak{m}_x$.

Then

> **Proposition:** Let X be a compact Hausdorff space. We have a homeomorphism
>
> $$\begin{aligned} X &\to \operatorname{maxSpec}\mathcal{C}(X)\\ x &\mapsto \mathfrak{m}_x\\ V(\mathfrak{m}):=\{x\in X|\forall f\in\mathfrak{m}, f(x)=0\} &\mapsfrom \mathfrak{m}\end{aligned}$$

and multiplying their values). For each $x \in X$, let $\mathfrak{m}_x$ be the set of all $f \in C(X)$ such that $f(x) = 0$. The ideal $\mathfrak{m}_x$ is maximal, because it is the kernel of the (surjective) homomorphism $C(X) \to \mathbf{R}$ which takes f to $f(x)$. If $\tilde{X}$ denotes Max $(C(X))$, we have therefore defined a mapping $\mu: X \to \tilde{X}$, namely $x \mapsto \mathfrak{m}_x$.

We shall show that μ is a homeomorphism of X onto $\tilde{X}$.

i) Let $\mathfrak{m}$ be any maximal ideal of $C(X)$, and let $V = V(\mathfrak{m})$ be the set of common zeros of the functions in $\mathfrak{m}$: that is,

$$V = \{x \in X : f(x) = 0 \text{ for all } f \in \mathfrak{m}\}.$$

Suppose that V is empty. Then for each $x \in X$ there exists $f_x \in \mathfrak{m}$ such that $f_x(x) \neq 0$. Since f_x is continuous, there is an open neighborhood U_x of x in X on which f_x does not vanish. By compactness a finite number of the neighborhoods, say $U_{x_1}, \ldots, U_{x_n}$, cover X. Let

$$f = f_{x_1}^2 + \cdots + f_{x_n}^2.$$

Then f does not vanish at any point of X, hence is a unit in $C(X)$. But this contradicts $f \in \mathfrak{m}$, hence V is not empty.

Let x be a point of V. Then $\mathfrak{m} \subseteq \mathfrak{m}_x$, hence $\mathfrak{m} = \mathfrak{m}_x$ because $\mathfrak{m}$ is maximal. Hence μ is surjective.

ii) By Urysohn's lemma (this is the only non-trivial fact required in the argument) the continuous functions separate the points of X. Hence $x \neq y \Rightarrow \mathfrak{m}_x \neq \mathfrak{m}_y$, and therefore μ is injective.

iii) Let $f \in C(X)$; let

$$U_f = \{x \in X : f(x) \neq 0\}$$

and let

$$\tilde{U}_f = \{\mathfrak{m} \in \tilde{X} : f \notin \mathfrak{m}\}$$

Show that $\mu(U_f) = \tilde{U}_f$. The open sets U_f (resp. $\tilde{U}_f$) form a basis of the topology of X (resp. $\tilde{X}$) and therefore μ is a homeomorphism.

Thus X can be reconstructed from the ring of functions $C(X)$.

Current note has 0 direct children and 0 total descendants.

- structures based on sets
  - Topological spaces
    - Types of topological spaces
      - Compact Hausdorff spaces
        - f R
          - reconstructing compact Hausdorff X from $\mathcal{C}(X,\mathbb{R})$

And it has 1 siblings.

- structures based on sets
  - Topological spaces
    - Types of topological spaces
      - Compact Hausdorff spaces
        - f R
          - reconstructing compact Hausdorff X from $\mathcal{C}(X,\mathbb{R})$