Info

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is written (completely with human hands) by Rupadarshi Ray,
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Pythagorean triples $Z(x^2+y^2-z^2)(\mathbb{N})$

Definition. Pythagorean triples

A **Pythagorean triple** is an element

$$(a,b,c)\in\mathbb{Z}_{\geq 0}^3$$

such that

$$a^2+b^2=c^2$$

Euclid's formula

- We calculate

$$\begin{aligned} c^2-a^2&=b^2\\ (c+a)(c-a)&=b^2\\ \frac{c+a}{b}&=\frac{b}{c-a}\\ &=\frac{m}{n}(\text{say, in lowest terms})\\ \frac{c}{b}+\frac{a}{b}=\frac{m}{n},\ \frac{c}{b}-\frac{a}{b}&=\frac{n}{m}\\ \frac{c}{b}=\frac{m^2+n^2}{2mn},\ \frac{a}{b}&=\frac{m^2-n^2}{2mn} \end{aligned}$$

- Let $a,b,c\in\mathbb{Z}_{>0}$ be copime and thus pairwise coprime. As a,b are coprime, at least one of them is odd, say a.
  - Then if b were also odd, c would be even, then $c^2 \bmod 4\equiv 0$ but

$$a^2+b^2\equiv 2 \bmod 4$$

    because odd square is congruent to 1 modulo 4. Thus b is even and c is odd.
  - Then in

$$\frac{c}{b}=\frac{m^2+n^2}{2mn},\ \frac{a}{b}=\frac{m^2-n^2}{2mn}$$

    as m/n is reduced, m,n are coprime so they cannot both be even.
  - ?

The Pythagorean triples a,b,c are of the form

$$\begin{aligned} a&=m^2-n^2\\ b&=2mn\\ c&=m^2+n^2 \end{aligned}$$

for any $(m,n)\in\mathbb{Z}_{>0}^2$, $m>n$.

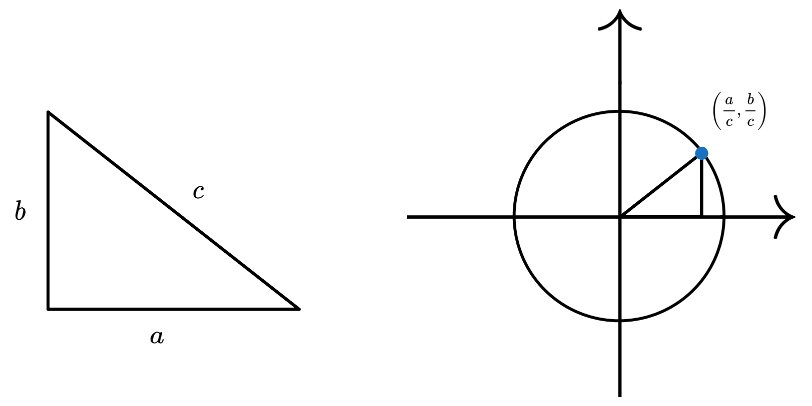
rational parameterization of circle

Intuition

As $c\neq 0$, a tiple $a,b,c\in\mathbb{Z}_{>0}$ is Pythagorean $\left(\frac{a}{c}\right)^2+\left(\frac{b}{c}\right)^2=1$

$$\Longleftrightarrow$$

$\left(\frac{a}{c},\frac{b}{c}\right)$ is a rational point on the circle, that is, an element of $V_{\mathbb{Q}}(X^2+Y^2-1)$



In particular, the primitive Pythagorean triples are bijective with rational points on the circle

$$\begin{aligned} \{\text{primitive Pythagorean triples}\}&\leftrightarrow V_{\mathbb{Q}}(X^2+Y^2-1)\\ (a,b,c)&\leftrightarrow\left(\frac{a}{c},\frac{b}{c}\right) \end{aligned}$$

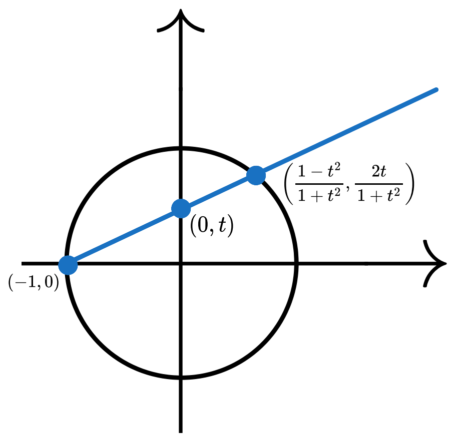
To find such points, we use the

Definition. Rational parameterization of the circle

The parameterization of the real circle using rational functions (rational parameterization)

$$\begin{aligned} \mathbb{R}&\to S^1=V_{\mathbb{R}}(X^2+Y^2-1)\\ t&\mapsto\left(\frac{1-t^2}{1+t^2},\frac{2t}{1+t^2}\right) \end{aligned}$$

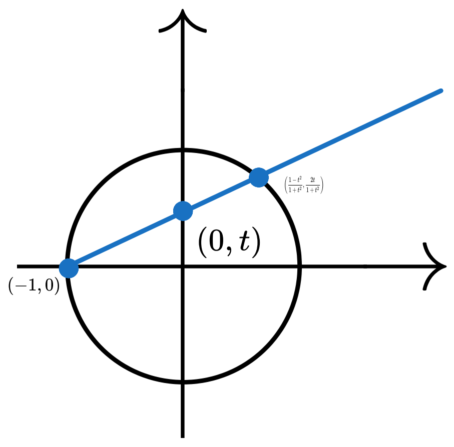
which is the point where the line passing through $(-1,0)$ and $(0,t)$ meets the circle



which has an inverse

$$(x,y) \mapsto t(x,y) := \frac{y}{x+1}$$

except for the point $(-1,0)$



In

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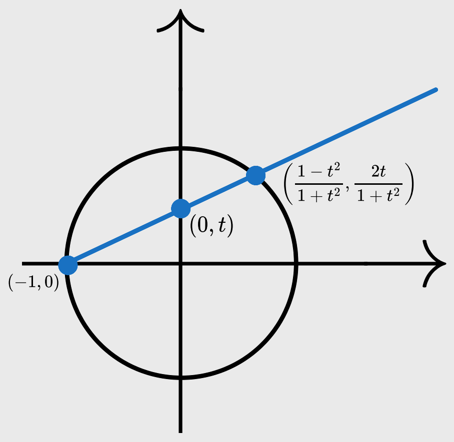

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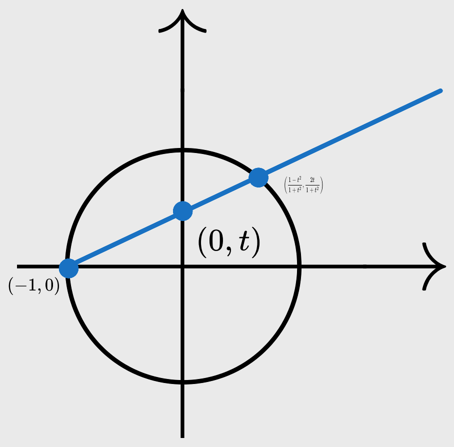

a rational t maps to a rational if

$$t(x,y) := \frac{y}{x+1} \in \mathbb{Q}$$

Current note has 0 direct children and 0 total descendants.

- squishy
  - Polynomials
    - Pythagorean triples

And it has 17 siblings.

- squishy
  - Polynomials
    - $X^n + Y^n = Z^n$
    - $8X^3 + 4X^2 - 4X - 1$
    - $x^2 + y^2 - 3$
    - Folium of Descartes

$$X^3 + Y^3 - 3aXY$$

    - $X^3 + Y^3 - a$
    - $3X^3 + 10X^2 + 3 - Y^2$
    - $3X^8 + 10X^4 + 3 - Y^2$
    - $XYZ - 1$
    - Pythagorean triples
    - Bolza curve

- Irreducible polynomials in $\mathbb{C}[x, y]$
- Polynomial maps $\mathbb{C}^n \to \mathbb{C}^n$ of degree at most d
- Chebyshev polynomials of the first kind
- Logistic map
- Dynamics of the map $z^2 + c$
- $\begin{pmatrix} x \\ y \\ y \end{pmatrix} \mapsto \begin{pmatrix} (1+xy)^3 z + y^2(1+xy)(4+3xy) \\ y + 3x(1+xy)^2 z + 3xy^2(4+3xy) \\ 2x - 3x^2 y - x^3 z \end{pmatrix}$
- $(w^2 - 1)^2$