Info

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Global algebraic functions on a complex 1-manifold

Let X be a Riemann surface and $p \in \mathscr{M}(X)[T]$ be a monic polynomial with coefficients in the field of meromorphic functions on X. That means

$$p(T) = T^n + c_1 T^{n-1} + \cdots + c_n$$

where c_i are meromorphic functions on X.

A *solution* to this polynomial must be a $F \in \mathscr{M}(X)$ such that $p(F) = 0$ that is

$$p(F) = F^n + c_1 F^{n-1} + \cdots + c_n = 0$$

everywhere on X.

In general, solutions to such polynomial equations would *not* exist. As in, $\mathscr{M}(X)$ are not algebraically closed in general.

Riemann surface of algebraic functions exist and they are unique

> **Proposition:**
> Suppose X is a Riemann surface and
>
> $$p(T) = T^n + c_1 T^{n-1} + \cdots + c_n \in \mathscr{M}(X)[T]$$
>
> is an *irreducible* polynomial of degree n.
> Then there **exists** a n-sheeted ($\implies$ proper [1]) holomorphic map and a meromorphic $F \in \mathscr{M}(Y)$
>
> $$\begin{tikzcd} Y \arrow[r, "F"'] \arrow[d, "\pi, \deg n"] & \mathbb{C}P^1 \\ X \end{tikzcd}$$
>
> such that $(\pi^* p)(F) = 0$ called the **algebraic function defined by the polynomial** $p(T)$.
> This $\pi : Y \xrightarrow{n} X, F$ is **unique** up to isomorphism, that is if there is another degree n map $Z \xrightarrow{n} X$
>
> $$\begin{tikzcd} Z \arrow[rr, bend left, "G"] \arrow[dr, "\deg n"'] & Y \arrow[r, "F"'] \arrow[d, "\pi, \deg n"] & \mathbb{C}P^1 \\ & X \end{tikzcd}$$
>
> then there is a biholomorphic map $\sigma : Z \to Y$ such that
>
> $$\begin{tikzcd} Z \arrow[r] \arrow[rr, bend left, "G"] \arrow[dr, "\deg n"'] & Y \arrow[r, "F"'] \arrow[d, "\pi, \deg n"] & \mathbb{C}P^1 \\ & X \end{tikzcd}$$
>
> commutes ($G = \sigma^* F$).
>
> Because the the map $\pi : Y \to X$ is proper, in particular if X is compact, then Y is also compact.

[2]

Let $\Delta \in \mathscr{M}(X)$ be the discriminant of the polynomial $p(T)$.

- p is irreducible $\implies \exists$ closed discrete $A \subset X$ such on $X \setminus A$ the functions $c_1, \ldots, c_n$ are holomorphic and $\Delta \neq 0$.
    - Then for $x \in X' := X \setminus A$

      $$p_x(T)$$

      has n distinct zeros.
- Now from the topological space $|\mathscr{O}_X| \to X$ we consider

  $$Y' := \left\{ f_x \in \mathscr{O} \middle| x \in X', p(f_x) = 0 \right\}$$

    - By [-] the projection $Y' \to X$ is a degree n covering.
- *Uniqueness* [3]

Current note has 0 direct children and 0 total descendants.

And it has 3 siblings.

- Non-constant holomorphic maps between Riemann surfaces locally looks like w^k and are open maps

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1. algebraic geometry - How to show that finite holomorphic maps are proper? - Mathematics Stack Exchange ↩
2. Otto Forster, Bruce Gilligan - Lectures on Riemann Surfaces (1991), p.60 ↩
3. Otto Forster, Bruce Gilligan - Lectures on Riemann Surfaces (1991), p.61 ↩