ⓘ Info

This note, that was found here, as a part of a collection
is written (completely with human hands) by Rupadarshi Ray,
was created on March 17, 2024 3:20:32 PM,
was last modified on September 7, 2026 1:11:12 PM, and
was exported on September 7, 2026 3:09:59 PM.

Non-constant holomorphic functions locally look like w^k and are open mappings

(Open mapping theorem for open subsets in $\mathbb{C}$) Let f be *non-constant* holomorphic function on a connected open set U then $f(U)$ is open (and connected).

non-constant holomorphic function locally looks like w^k on a change of coordinates

Let we have a holomorphic function f on some open subset of $\mathbb{C}$ containing z_0.

> **Proposition:** If $f(z_0) \neq 0$ and *any* positive integer k, we compose f with the function $z^{1/k}$ to obtain a function h such that
>
> $$h^k = f \text{ on } V$$
>
> for a smaller open V containing z_0.

> **Proposition:** If $f : U \to \mathbb{C}$ is a *non-constant* holomorphic function on a open subset $U \subseteq \mathbb{C}$,
>
> - then around any $z_0 \in U$ there is a holomorphic $g(z)$
>
> $$\begin{aligned} f(z) = f(0) + (z - z_0)^k g(z) \text{ around } z_0 \\ g(z_0) \neq 0 \end{aligned}$$
>
> - Moreover there is a smaller neighborhood around z_0 and a holomorphic $h(z)$ such that
>
> $$\begin{aligned} f(z) = f(0) + ((z - z_0)h(z))^k \text{ around } z_0 \\ h(z_0) \neq 0 \end{aligned}$$
>
> with further smaller neighborhood around z_0 so that $(z - z_0)h(z)$ is a biholomorphism to its (open) image.

- Now, we write

$$f(z) = \sum_{n \geq 0} a_n (z - z_0)^n \text{ on } z \in B_R(z_0)$$

- If f is **not constant**, there is some smallest $k \geq 1$ (**multiplicity of the zero**) such that $a_k \neq 0$, meaning

$$\begin{aligned} f(z) = a_0 + (z - z_0)^k g(z) \text{ on } z \in B_R(z_0) \\ g(z) = \sum_{n \geq 0} a_{n+k} (z - z_0)^n \\ g(z_0) \neq 0 \end{aligned}$$

- Then by

> **Proposition:** If $f(z_0) \neq 0$ and *any* positive integer k, we compose f with the function $z^{1/k}$ to obtain a function h such that
>
> $$h^k = f \text{ on } V$$
>
> for a smaller open V containing z_0.

we find a smaller open V containing z_0 and a holomorphic h such that

$$\begin{aligned} f(z) = a_0 + (z - z_0)^k (h(z))^k \text{ on } z \in V \\ h(z_0) \neq 0 \end{aligned}$$

- Thus we have

$$f(z) = a_0 + (\underbrace{(z - z_0)h(z)}_{h_1(z)})^k \text{ on } z \in V$$

with $h(z_0) \neq 0$, so

$$\begin{aligned} h_1(z) &= (z - z_0)h(z) \\ h_1'(z) &= h(z) + (z - z_0)h'(z) \\ h_1'(z_0) &= h(z_0) \neq 0 \end{aligned}$$

- As h_1 has a non-vanishing derivative at z_0, so there must be *another* smaller open W containing z_0 where $h_1(W)$ is open and

$$h_1 : W \to h_1(W)$$

is a homeomorphism.
- We can choose W so that $h_1(W)$ is a disk centered at $h_1(z_0) = 0$.

> **Proposition:**
>
> $$\begin{aligned} f : W \to f(W) = a_0 + D \\ f(z) = a_0 + (h_1(z))^k \end{aligned}$$
>
> is an **open** map.

- Given open set $A \subseteq W$ its image $h_1(A)$ under the homeomorphism h_1 is open set in a disk containing 0.
- Image of $h_1(A)$ under $z \mapsto z^k$ is open and translation $w \mapsto a_0 + w$ takes open sets to open sets.

- Thus $f(A)$ is open.

holomorphic functions to closed sets must map to interior

Let

$$f:\Omega\to\bar{U}$$

is a *non-constant* holomorphic function from a open $\Omega\subseteq\mathbb{C}$ to a closed set $\bar{U}$ with open interior U.

Then by

> **(Open mapping theorem for open subsets in $\mathbb{C}$)** Let f be *non-constant* holomorphic function on a connected open set U then $f(U)$ is open (and connected).

we must have

$$f(\Omega)\subseteq U$$

Even, by (the statement of) maximum modulus we may conclude the same for function

$$f:\Omega\to B_R(0)$$

going to a ball centered at 0: if there were a point in the image which is in the boundary of the ball

$$p\in U, f(p)\in f(\Omega)\cap S^1_R$$

then p would be a maxima for $|f|$.

This has implications:

- Let

$$f_n:\Omega\to U$$

be a sequence of functions converging to f uniformly on compact subsets of Ω. By

> Let $f_n:U\to\mathbb{C}$ be holomorphic on open $U\subseteq\mathbb{C}$ and
>
> $$f_n\xrightarrow{\text{uniformly as } n\to\infty} f \text{ on } K$$
>
> *for every $K\subset U$ compact* then f is holomorphic in U.

the limit f is holomorphic. Then as $f_n(z)\in U$ the limit must be in $\bar{U}$, but because image must be open, so we conclude the limit is a holomorphic

$$f:\Omega\to U$$

  - So we have

$$z\in\Omega\implies\lim_{n\to\infty}f_n(z)\in U$$

even though $f_n(z)\in U$ and U is open.

Current note has 0 direct children and 0 total descendants.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $_\Gamma\backslash^{\mathbb{R}^n}, {}_\Gamma\backslash^{\mathbb{C}^n}$
    - Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
      - mapping
        - Non-constant holomorphic functions locally look like w^k and are open mappings

And it has 13 siblings.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $_\Gamma\backslash^{\mathbb{R}^n}, {}_\Gamma\backslash^{\mathbb{C}^n}$
    - Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
      - mapping
        - Bounding derivative of holomorphic mappings
        - Bounding holomorphic functions linearly
        - Branch points and ramification index of holomorphic functions
        - Disk in holomorphic image of disk
        - Size of holomorphic image of disks
        - Maximum and minimum of holomorphic functions
        - Fibers of holomorphic mappings
        - Non-constant holomorphic functions locally look like w^k and are open mappings
        - Biholomorphic mapping onto disk
        - Conformal radius of pointed simply connected open subsets of $\mathbb{C}$
        - Holomorphic functions on punchered disk
        - Holomorphic function whose singularities are not isolated
        - A holomorphic map branched over $\mathbb{Q}$