T-closed Lie subalgebras are reductive

Proposition 1.59. Let $\mathfrak{g}$ be a real Lie algebra of matrices over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. If $\mathfrak{g}$ is closed under the operation conjugate transpose, then $\mathfrak{g}$ is reductive.

REMARK. We write $(\cdot)^*$ for conjugate transpose.

PROOF. For matrices X and Y, define $\langle X, Y\rangle = \operatorname{Re}\operatorname{Tr}(XY^*)$. This is a real inner product on $\mathfrak{g}$, the symmetry following from (1.58). Let $\mathfrak{a}$ be an ideal in $\mathfrak{g}$, and let $\mathfrak{a}^\perp$ be the orthogonal complement of $\mathfrak{a}$ in $\mathfrak{g}$. Then

$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$ as vector spaces. To see that $\mathfrak{a}^\perp$ is an ideal in $\mathfrak{g}$, let X be in $\mathfrak{a}^\perp$, let Y be in $\mathfrak{g}$, and let Z be in $\mathfrak{a}$. Then

$$\begin{aligned}\langle [X, Y], Z\rangle &= \operatorname{Re}\operatorname{Tr}(XYZ^* - YXZ^*)\\ &= -\operatorname{Re}\operatorname{Tr}(XZ^*Y - XYZ^*) \qquad \text{by (1.58)}\\ &= -\operatorname{Re}\operatorname{Tr}(X(Y^*Z)^* - X(ZY^*)^*)\\ &= -\langle X, [Y^*, Z]\rangle.\end{aligned}$$

Since Y^* is in $\mathfrak{g}$, $[Y^*, Z]$ is in $\mathfrak{a}$. Thus the right side is 0 for all Z, and $[X, Y]$ is in $\mathfrak{a}^\perp$. Hence $\mathfrak{a}^\perp$ is an ideal, and $\mathfrak{g}$ is reductive.

Current note has 0 direct children and 0 total descendants.

- stem
  - $GL(n)$
    - T, †-closed subgroups of $GL(n)$
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And it has 1 siblings.

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