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Smooth action by a compact Lie group

upper bound of dimension

Let a **compact Lie group** G act smoothly, faithfully on a *connected* smooth manifold M. Then

$$\dim G \leq \frac{1}{2}\dim M(\dim M + 1)$$

(5.17) Proposition. *Let the compact Lie group G act differentiably and effectively on the connected n-dimensional manifold M. Then* $\dim G \leqq \frac{1}{2}n(n+1)$.

Proof. (Induction on n.) The case $n = 0$ is trivial. We may assume that G is connected because the component of e acts effectively and has the same dimension. Let G/H be the principal orbit type of M. Then $\dim G \leqq n + \dim H$. If we knew that H acts effectively on a manifold of dimension at most $(n-1)$, then we could use the induction hypothesis to prove the theorem for n because $n + \frac{1}{2}(n-1)n = \frac{1}{2}n(n+1)$. We have $\dim G/H = k \leqq n$ and G acts effectively on G/H; in fact, if $g \in G$ acts as identity on G/H, then it acts as identity on any principal orbit and hence on M. Thus H_0, the component of e in H, acts effectively on G/H and on the principal orbit of the H_0-manifold G/H which is connected. If the dimension of this principal orbit is less than k, we are done. Otherwise, $H_0eH = G/H$ and G/H is a point; then $G = \{e\}$ since it acts effectively. □

Manifolds for which equality holds in (5.17) are to be considered the most symmetric ones. The group $O(n+1)$ of dimension $\frac{1}{2}n(n+1)$ acts effectively on the n-sphere and on the n-dimensional real projective space. No wonder that mathematicians like these spaces (see (2.12)).

- *proof by constructing Riemannian metric*

specific constraints

- The only compact connected Lie group that can act faithfully on a torus is itself
- There are smooth manifolds which does not admit action by any compact Lie group [1] [2]

Current note has 0 direct children and 0 total descendants.

- structures based on sets
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      - Smooth Lie group action on a smooth manifold
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And it has 5 siblings.

- structures based on sets
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        - Orbits of smooth Lie group actions
        - Proper Lie group actions
        - invariant Riemannian metric in a proper Lie group action

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1. Examples of asymmetric differentiable manifolds | Mathematische Annalen ↩
2. algebraic topology - Does a topological space always admit a $\mathbb{Z}_2$-action? - Mathematics Stack Exchange ↩