Info

This note, that was found here, as a part of a collection
is written (completely with human hands) by Rupadarshi Ray,
was created on June 4, 2025 6:45:00 PM,
was last modified on August 30, 2026 3:39:44 PM, and
was exported on September 7, 2026 3:09:46 PM.

Fuchsian groups $\Gamma <_{\mathsf{d}} PSL(2)(\mathbb{R})$

$\leftrightarrow$

oriented hyperbolic 2-manifold/orbifolds

$\leftrightarrow$

convex polygons in $\mathbb{R}\boldsymbol{H}^2$ equipped with a cocomplete, subproper $\mathrm{Isom}(\mathbb{R}\boldsymbol{H}^2)_\circ$-side-pairing

$\rightarrow$

Riemann surfaces with branching signature

Definition. Definition

A **hyperbolic 2-manifold** is a complete, connected Riemannian 2-manifold with curvature $\kappa \equiv -1$.
A **hyperbolic 2-orbifold** is a quotient ${}_{\Gamma}\backslash^{\mathbb{R}\boldsymbol{H}^2}$ where $\Gamma < \mathrm{Isom}(\mathbb{R}\boldsymbol{H})$ is a discrete subgroup.

Definition. A **Fuchsian group** is a discrete subgroup of $PSL(2)(\mathbb{R})$.

We know $\Gamma < PSL(2)(\mathbb{R})$ is a Fuchsian group $\iff$ the action $\Gamma \curvearrowright \mathbb{R}H^2$ is finitely discontinuous.

Dirichlet domain

Dirichlet domain of group acting isometrically and finitely discontinuously

> **Proposition:** Let Γ be Fuchsian, $H \leq \Gamma$ be of finite index and F be a proper fundamental domain of Γ. Then
>
> $$D = \bigcup_{\gamma \in \Gamma/H} \gamma F$$
>
> is a proper fundamental domain of Γ.
> Moreover,
>
> $$\mu(D) = [\Gamma : H]\mu(F)$$
>
> if $\mu(F) < \infty$.

> **Proposition:** Let Γ is a Fuchsian group and F is a fundamental domain for Γ.
>
> - If $\gamma \in \Gamma$ is elliptic then $g\gamma g^{-1}$ fixes a point in F for some $g \in \Gamma$.
> - If $z \in \mathrm{int}(F)$ then $\Gamma_z = \{I\}$
>
> In particular, $H \leq \Gamma$ is a nontrivial finite subgroup then $gHg^{-1} \leq \Gamma_z$ for some $z \in \partial F$ and $g \in \Gamma$.

Let $\gamma z_0 = z_0$ and $gz_0 \in \Gamma$ Then

$$g\gamma g^{-1}(gz_0) = gz_0$$

sides and vertices of a Dirichlet domain

As

$$\partial D(\Gamma, z_0) \subseteq \bigcup_{\gamma \in \Gamma} L_\gamma(z_0)$$

Definition. Sides and vertices of a Dirichlet domain

We have the set of geodesic segments that bounds $D(\Gamma)$

$$\mathrm{sides}(D(\Gamma))$$

where $\bigcup \mathrm{sides}(D(\Gamma)) = \partial D(\Gamma)$ and the vertices of $D(\Gamma)$ are

$$\mathrm{vertex}(D(\Gamma)) \subseteq D$$

The set of **ideal vertices** is

$$\partial_\infty D(\Gamma)$$

The set of vertex points

$$\mathrm{vertex}(D(\Gamma)) \subseteq \partial D$$

is a closed, discrete set.

For all $z \in D(\Gamma, z_0)$ we have

$$|\Gamma\{z\} \cap \partial D_\Gamma(z_0)| < \infty$$

and

$$\sum_{z\in\Gamma\{z\}\cap\partial D_\Gamma(z_0)} \angle_z D_\Gamma(z_0) = \frac{2\pi}{|\Gamma_z|}$$

$$\{z \in D(\Gamma) | \Gamma_z \neq \{I\}\} \subseteq \mathrm{vertex}(D) \cup \mathrm{mid}(\mathrm{sides}(D(\Gamma)))$$

side parings

A Fuchsian group Γ is generated by side pairing maps, that is, the image of

$$\begin{aligned}\mathrm{sides}(D(\Gamma)) \times \mathrm{sides}(D(\Gamma)) &\to \Gamma \cup \{\emptyset\}\\ (\sigma_1, \sigma_2) &\mapsto \gamma \in \Gamma \setminus \{I\} : \gamma(\sigma_1) = \sigma_2\end{aligned}$$

cyclic subgroups

$$\{z \in H^2 \big| \Gamma_z \neq \{I\}\} \leftrightarrow \{\text{maximal finite/elliptic cyclic } H \leq \Gamma\}$$

Current note has 17 direct children and 18 total descendants.

- squishy
  - Polynomials
    - Bolza curve
- stem
  - Objects stemming from $\mathbb{R}$-spaces of dimension 2
  - Objects stemming from $PSL(2)(\mathbb{R})$
    - Fuchsian groups $\Gamma <_d PSL(2)(\mathbb{R})$ $\leftrightarrow$ oriented hyperbolic 2-manifold/orbifolds $\to$ Riemann surfaces with branching signature
      - Artihmetic Fuchsian groups and Artihmetic hyperbolic 2-orbifolds
      - Cocompact Fuchsian groups
        - Heat flow of a cocompact Fuchsian group
      - Central simple algebra
      - Elementary Fuchsian groups
      - Finitely generated Fuchsian groups $\leftrightarrow$ geometrically finite Fuchsian groups
      - Fuchsian groups of finite coarea
      - Geodesic flow of Fuchsian groups and hyperbolic surfaces
      - Decomposition of hyperbolic surfaces
      - Infinitely generated Fuchsian groups
      - Limit set of Fuchsian groups
      - Marked hyperbolic surfaces
      - subst.R PSL 2 discrete subg.over Q-bar
      - Fuchsian Schottky groups
      - Spectrum of hyperbolic surfaces
      - Torsion-free Fuchsian groups
      - Cocompact torsion-free Fuchsian groups
      - Fuchsian triangle groups

And it has 3 siblings.

- stem
  - Objects stemming from $PSL(2)(\mathbb{R})$
    - $\mathrm{HomGrp}(F_2, PSL(2)(\mathbb{R}))$
    - 2-generated subgroups of $PSL(2)(\mathbb{R})$
    - Fuchsian groups $\Gamma <_d PSL(2)(\mathbb{R})$ $\leftrightarrow$ oriented hyperbolic 2-manifold/orbifolds $\to$ Riemann surfaces with branching signature