Info

This note, that was found here, as a part of a collection
is written (completely with human hands) by Rupadarshi Ray,
was created on March 9, 2026 10:28:24 PM,
was last modified on September 2, 2026 11:27:52 AM, and
was exported on September 7, 2026 3:10:00 PM.

Polyhedron gluings

Definition. Convex sets in $\mathbb{R}^n, \mathbb{R}\boldsymbol{H}^n$ or S^n

A subset $C \subseteq \mathbb{R}^n, \mathbb{R}\boldsymbol{H}^n$ or $S^n, n > 0$ is **convex** if for every pair of points $x, y \in C$ such that $d(x, y) < \operatorname{diam}(\mathbb{R}^n, \mathbb{R}\boldsymbol{H}^n$ or $S^n)$, the (unique) geodesic segment between x, y is contained in C

$$\overline{xy} \subseteq C$$

We have

- the **dimension** of C

$$\dim C := \min\{m \in \mathbb{N} | C \text{ is contained in an } m\text{-plane}\}$$

- $\operatorname{plane}(C)$ is the unique $\dim C$-plane that contains C
- a **side** of C is a non-empty, maximal convex subset of ∂C.

$$\operatorname{sides}(C) := \max(\{\emptyset \neq A \subseteq \partial C | A \text{ is convex}\}, \subseteq)$$

  is the (possibly uncountable) set of all sides of C

Definition. Convex polyhedron

A **convex polyhedron** P in $\mathbb{R}^n, \mathbb{R}\boldsymbol{H}^n$ or S^n is a non-empty, closed, (geodesically) convex subset of $\mathbb{R}^n, \mathbb{R}\boldsymbol{H}^n$ or S^n such that the collection $\operatorname{sides}(P)$ of sides of P is locally finite.

side pairing

Definition. Side pairing of convex polyhedra

Let $G \leq \operatorname{Isom}(\mathbb{R}^n, \mathbb{R}\boldsymbol{H}^n$ or $S^n)$ and $\mathcal{P}$ be a finite collection of disjoint convex polyhedra in $\mathbb{R}^n, \mathbb{R}\boldsymbol{H}^n$ or S^n. A G-**side pairing** for $\mathcal{P}$ is a collection

$$\Phi = \{g_S | S \in \operatorname{sides}(\mathcal{P})\}$$

such that

- *(side pairing)* for each side $S \in \operatorname{sides}(\mathcal{P})$ there is a side $S' \in \operatorname{sides}(\mathcal{P})$ such that $S \overset{g_S}{\longmapsto} S'$ (here, S is saied to be **paired** with S')
- *(symmetry of paired sides)* for each $S \in \operatorname{sides}(\mathcal{P})$ we have $g_{S'}^{-1} = g_S$
- *(tesselation)* for each $P \in \mathcal{P}, S \in \operatorname{sides}(P)$, which is *paired* with $S' \in \operatorname{sides}(P')$, we have

$$P \cap g_S(P') = S'$$

Here,

- The pairing of sides by Φ generates an equivalence relation on $\bigcup \operatorname{sides}(\mathcal{P})$. An equivalence class $[S] \in {}_{\Phi}\backslash \bigcup \operatorname{sides}(\mathcal{P})$ is called a **cycle** of Φ.
- The **angle** at $x \in P$ is

$$\angle_x P := \frac{4\pi\mu(P \cap B(r, x))}{\mu(B(x, r))}$$

  for any $r < \min_{S \in \operatorname{sides}(\mathcal{P}, x \notin S)} d(x, S)$ [1]
- Let $[x] = \{x_1, \ldots, x_m\} \in {}_{\Phi}\backslash \bigcup \operatorname{sides}(\mathcal{P})$ be a finite cycle of Φ where $x_l \in P_l$ for each $1 \leq l \leq m$. The **solid angle sum** at $[x]$ is

$$\angle_{[x]} P := \sum_l \angle_{x_l} P_l$$

We deduce

- $(S')' = S$

Definition. Proper side pairing

A G-side pairing Φ for $\mathcal{P}$ is **proper** if each cycle of Φ is finite and has solid angle 4π.

Theorem 10.1.3. *Let G be a group of orientation-preserving isometries of X and let $\Phi = \{g_S : S \in \mathcal{S}\}$ be a G-side-pairing for a finite family $\mathcal{P}$ of disjoint convex polyhedra in X. Then Φ is proper if and only if*

(1) *each cycle of Φ is finite,*
(2) *the isometry g_S fixes no point of S' for each S in $\mathcal{S}$,*
(3) *each edge cycle of Φ has dihedral angle sum 2π.*

Theorem 10.1.1. *If G is a group of isometries of X and Φ is a proper G-side-pairing for a finite family $\mathcal{P}$ of disjoint convex polyhedra in X, then*

(1) *the isometry g_S fixes no point of S' for each S in $\mathcal{S}$,*

(2) *the sides S and S' are equal if and only if S is a great 2-sphere of S^3 and g_S is the antipodal map of S^3,*

(3) *each edge cycle of Φ contains at most one point of an edge of a polyhedron in $\mathcal{P}$.*

Theorem 10.1.2. *Let G be a group of isometries of X and let M be the space obtained by gluing together a finite family $\mathcal{P}$ of disjoint convex polyhedra in X by a proper G-side-pairing Φ. Then M is a 3-manifold with an (X, G)-structure such that the natural injection of P° into M is an (X, G)-map for each P in $\mathcal{P}$.*

complete gluing of finite sided polyhedra

Theorem 11.1.2. *Let M be a Euclidean n-manifold obtained by gluing together a finite family $\mathcal{P}$ of disjoint, finite-sided, n-dimensional, convex polyhedra in E^n by a proper $\mathrm{I}(E^n)$-side-pairing Φ. Then M is complete.*

Definition. Cusp points of a convex polyhedron

Let $P \subseteq \mathbb{R}\boldsymbol{H}^n$ be a convex polyhedron. A **cusp point** of P is an ideal point $c \in \partial_\infty P$ for which there is an onb $N_c \subseteq \overline{\mathbb{R}\boldsymbol{H}^n}$ such that

$$\bigcap_{S \in \mathrm{sides}(P) : c \in \partial_\infty S, S \cap N_c \neq \emptyset} \overline{S}^\infty = \{c\}$$

Definition. Link of a cusp of a polyhedron

Let $\mathcal{P}$ be a finite collection of disjoint, n-dimensional, convex polyhedra in $\mathbb{R}\boldsymbol{H}^n$ and Φ be a proper $\mathrm{Isom}(\mathbb{R}\boldsymbol{H}^n)$-side pariring for $\mathcal{P}$. We assume without loss of generality that no two distinct polyhedrons in $\mathcal{P}$ have closures that intersect in $\overline{\mathbb{R}\boldsymbol{H}^n}$ and that no polyhedron passes through the point $\infty \in \overline{H^n_{\mathsf{U}}}$.

- Then Φ extends to a side pairing of $\left\{\overline{P}^\infty \middle| P \in \mathcal{P}\right\}$ which generates an equivalence relation on the closure of the sides, whose equivalence classes

$$[x] \in {}_{\Phi}\backslash^{\left\{\overline{S}^\infty \middle| S \in \mathrm{sides}(\mathcal{P})\right\}}$$

  are called **cycles**.
- The cycle of a *cusp point* of a polyhedron $P \in \mathcal{P}$ is called a **cusp point** of the quotient ${}_{\Phi}\backslash^{\bigcup \mathcal{P}}$.
- Let c be a cusp point of a polyhedron in $\mathcal{P}$ and $b \in [c]$. Then $\exists P_b \in \mathcal{P}$ such that $b \in \overline{P_b}^\infty$. The **link** of b is the $n-1$-dimensional convex polyhedron $L(b)$ in $\mathbb{R}^{n-1}$ obtained by intersecting P_b with a horosphere Σ_b based at b that meets just the sides of P_b incident with b.
- We choose the horospheres $\{\Sigma_b | b \in [c]\}$ small enough so that the links $L(b)$ for points $b \in [c]$ are mutually disjoint. Then Φ determines a *proper* $\mathrm{Sim}(\mathbb{R}^{n-1})$-side pairing for $\{L(b) | b \in [c]\}$

  Let g_S be an element of Φ and let u, u' be elements of $[v]$ such that $g_S(u') = u$. Then $\Sigma_{u'} \cap S'$ is a side of $L(u')$ and $\Sigma_u \cap S$ is a side of $L(u)$. Now let

$$\overline{g}_S : \Sigma_{u'} \to g_S(\Sigma_{u'})$$

  be the restriction of g_S. Then $\overline{g}_S$ is an isometry with respect to the natural Euclidean metrics of the horospheres $\Sigma_{u'}$ and $g_S(\Sigma_{u'})$. Observe that the line segment

$$g_S(\Sigma_{u'} \cap S') = g_S(\Sigma_{u'}) \cap S$$

  is parallel to the line segment $\Sigma_u \cap S$ because $g_S(\Sigma_{u'})$ is concentric with Σ_u. Let

$$p_S : g_S(\Sigma_{u'}) \to \Sigma_u$$

  be the radial projection of $g_S(\Sigma_{u'})$ onto Σ_u. Then p_S is a change of scale with respect to the natural Euclidean metrics of $g_S(\Sigma_{u'})$ and Σ_u. Define

$$h_S : \Sigma_{u'} \to \Sigma_u$$

  by $h_S = p_S \overline{g}_S$. Then h_S is a similarity with respect to the natural Euclidean metrics of $\Sigma_{u'}$ and Σ_u. Moreover h_S maps the side $\Sigma_{u'} \cap S'$ of $L(u')$ onto the side $\Sigma_u \cap S$ of $L(u)$. Clearly $\{h_S\}$ is a proper $\mathrm{S}(E^2)$-side-pairing of the polygons $\{L(u)\}$. Here S ranges over the set of all the sides in $\mathcal{S}$ incident with the cycle $[v]$. We will assume that the horospheres $\{\Sigma_u\}$ have been chosen so that $p_S = 1$ for the largest possible number of sides S.
- Let $L[c] := {}_{\Phi}\backslash^{\bigcup_{b \in [c]} L(b)}$ be the **link of the cusp point** $[c]$ of ${}_{\Phi}\backslash^{\bigcup \mathcal{P}}$.

> **Proposition:** The link $L[c]$ of a cusp point $[c]$ of ${}_{\Phi}\backslash^{\bigcup \mathcal{P}}$ is a connected $(\mathbb{R}^{n-1}, \mathrm{Sim}(\mathbb{R}^{n-1}))$-manifold.

Let M be a $(\mathbb{R}\boldsymbol{H}^n, \mathrm{Isom}(\mathbb{R}\boldsymbol{H}^n))$-manifold obtained by gluing together a finite faimly of disjoint, finite sided, n-dimensional, convex polyhedra in $\mathbb{R}\boldsymbol{H}^n$ by a proper $\mathrm{Isom}(\mathbb{R}\boldsymbol{H}^n)$-side pairing Φ. Then M is complete

$$\iff$$

$L[c]$ is complete for each cusp point $[c]$ of M

$$\iff$$

for each cusp point $[c]$, the links $\{L(b) | b \in [c]\}$ can be chosen so that Φ restricts to a side-pairing for them.

gluing

(Poincare's fundamental polyhedron theorem for discrete subgroups of $\mathrm{Isom}(\mathbb{R}^n, \mathbb{R}\boldsymbol{H}^n \text{ or } S^n)$) Let P be a n-dimensional convex polyhedron in $X \in \{\mathbb{R}^n, \mathbb{R}\boldsymbol{H}^n, S^n\}$ and Φ be a subproper $\mathrm{Isom}(X)$-side-pairing for P such that

$$\langle\Phi\rangle \backslash P$$

is complete. Then $\Gamma := \langle\Phi\rangle < \mathrm{Isom}(X)$ is discrete and, P becomes an exact, convex, fundamental polyhedron for Γ, the inclusion map $\iota : P \hookrightarrow X$ extends to an isometry

$$\Gamma \backslash P \cong_{\mathsf{Met}} \Gamma \backslash X$$

and

$$\begin{aligned}\langle \mathrm{sides}(P) | \text{side-pairings, cycle relations}\rangle &\cong_{\mathsf{Grp}} \Gamma \\ S &\mapsto g_S\end{aligned}$$

If Φ is proper then Γ acts freely on X.

Current note has 0 direct children and 0 total descendants.

- stem
    - Convex sets in $\mathbb{R}^n, \mathbb{R}\boldsymbol{H}^n$ or S^n
        - Polyhedron gluings

And it has 2 siblings.

- stem
    - Convex sets in $\mathbb{R}^n, \mathbb{R}\boldsymbol{H}^n$ or S^n
        - Polyhedron gluings
        - κ-simplicial complex

---

1. It follows from Theorems 2.4.1 and 3.4.1 that $\omega(P, x)$ does not depend on the radius r. ↩