

 Info

This note, that [was found here](#), as a part of [a collection](#) is written (completely with human hands) by [Rupadarshi Ray](#), was created on March 7, 2026 12:33:05 PM, was last modified on September 7, 2026 1:10:47 PM, and was exported on September 7, 2026 3:09:50 PM.

Integral on bounded open subsets of $\mathbb{R}^n$ with $\mathcal{C}^1$ boundary

object	In Cartesian coordinates
Volume form $\lambda_{\mathbb{R}^n} \equiv \lambda$	$dx_1 \wedge \cdots \wedge dx_n$

integrating *gradient vector field* $\otimes$ *volume form* in $\mathbb{R}^n$

 Let $B \subseteq \mathbb{R}^n$ be a bounded open set with $\mathcal{C}^1$ boundary. Then

$$\int_B \text{grad}(u) \otimes \lambda_B = \int_{\partial B} u \, \hat{n}_{\partial B} \otimes \lambda_{\partial B}$$

using **Stroke's theorem**

- Let $B \subset \mathbb{R}^n$ be a bounded open set with $\mathcal{C}^1$ boundary. Let $u \in \mathcal{C}^1(\overline{B})$ and consider

$${}^{n-1}\alpha = \widehat{u dx_i}$$

where $\widehat{dx_i}$ is $\lambda = dx_1 \wedge \cdots \wedge dx_n$ with dx_i removed.

- The **exterior derivative** of this $n - 1$ form is the n -form

$$d\alpha = \frac{\partial u}{\partial x_i} dx_i \wedge \widehat{dx_i} = (-1)^{i+1} \frac{\partial u}{\partial x_i} \lambda$$

where

$$\begin{aligned} dx_i \wedge \widehat{dx_i} &= dx_i \wedge dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_n \\ &= (-1)^{i+1} \underbrace{dx_1 \wedge \cdots \wedge dx_n}_{\lambda} \end{aligned}$$

- Then by *Stroke's theorem*

$$\begin{aligned} \int_B d\alpha &= (-1)^{i+1} \int_{\partial B} \alpha \\ \implies \int_B \frac{\partial u}{\partial x_i} \lambda &= (-1)^{i+1} \int_{\partial B} u \widehat{dx_i} \end{aligned}$$

 Example

For $n = 2$ we have

$$\int_{\partial B} -u dx = \int_B \frac{\partial u}{\partial y} dx \wedge dy$$

- Let us parameterize ∂B by a $\mathcal{C}^1$ function

$$\begin{aligned} U(\text{open}) &\subseteq \mathbb{R}^{n-1} \rightarrow B \\ y &\mapsto (g(y), y) \end{aligned}$$

where $g \in \mathcal{C}^1(U, \mathbb{R})$.

- Then

$$\begin{aligned} \int_{\partial B} u \widehat{dx_i} &= \int_U u(g(y), y) dg \wedge \widehat{dy_{i-1}} \\ &= \int_U u(g(y), y) \frac{\partial g}{\partial y_{i-1}} dy_{i-1} \wedge \widehat{dy_{i-1}} \\ &= (-1)^i \int_U u(g(y), y) \frac{\partial g}{\partial y_{i-1}} dy_1 \wedge \cdots \wedge dy_{n-1} \end{aligned}$$

for $i > 1$ and

$$= \int_U u(g(y), y) dy_1 \wedge \cdots \wedge dy_{n-1}$$

for $i = 1$.

- Thus

$$\int_B \frac{\partial u}{\partial x_i} \lambda = \begin{cases} \int_U u(g(y), y) dy_1 \wedge \cdots \wedge dy_{n-1} & i = 1 \\ - \int_U u(g(y), y) \frac{\partial g}{\partial y_{i-1}} dy_1 \wedge \cdots \wedge dy_{n-1} & i < n \end{cases}$$

- Tensoring with $\hat{e}_i$ and summing for $1 \leq i \leq n$ we have

$$\int_B \text{grad}(u) \otimes \lambda = \int_U u(1, -\text{grad}(g)) \otimes (dy_1 \wedge \cdots \wedge dy_{n-1})$$

where $X \otimes \lambda$ is a vector field $\otimes$ n -form, which we understand is integrated component-wise.

- We identify

$$(1, -\text{grad}(g)) \otimes (dy_1 \wedge \cdots \wedge dy_{n-1}) =: \hat{n}_{\partial B} \otimes \lambda_{\partial B}$$

then we have

$$\int_B \text{grad}(u) \otimes \lambda_B = \int_{\partial B} u \, \hat{n}_{\partial B} \otimes \lambda_{\partial B}$$

integration by parts in $\mathbb{R}^n$

☰ (Integration by parts in $\mathbb{R}^n$) Let $u, v \in \mathcal{C}^1(\overline{U})$. Then

$$\int_U u_{x_i} v = - \int_U u v_{x_i} + \int_{\partial U} u v \hat{n}_i$$

integral of divergence

☰ (Integral of divergence) Let $u \in \mathcal{C}^2(\overline{U})$. Then



$$\int_U \Delta u = \int_{\partial U} \frac{\partial u}{\partial \hat{n}}$$

🌀 Intuition

The Laplacian Δu measures how much $\text{grad}(u)$ moves points outside of U , however it must "move it" through the boundary by amount $\text{grad}(u) \cdot \hat{n}$ therefore

☰ (Integral of divergence) Let $u \in \mathcal{C}^2(\overline{U})$. Then



$$\int_U \Delta u = \int_{\partial U} \frac{\partial u}{\partial \hat{n}}$$

Current note has 0 direct children and 0 total descendants.

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And it has 4 siblings.

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