ⓘ Info

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Connection on a smooth vector bundle on a smooth manifold

Definition. Connection on a smooth vector bundle on a smooth manifold

Let $E \to M$ be a smooth $\mathbb{R}$-vector bundle over a smooth manifold M with the space of sections ΓE. A **connection** on $E \to M$ is a $\mathbb{R}$-linear map

$$\nabla : \Gamma E \to \Gamma(T^*M \otimes E)$$

such that

$$\nabla(f\phi) = \mathrm{d}f \otimes \phi + f\nabla(\phi)$$

for all $f \in \mathcal{C}^\infty(M), \phi \in \Gamma E$.

$$\nabla : \Omega^0(M, E) \to \Omega^1(M, E)$$

locally

On a trivialization U_α of $E \to M$, the section $\phi : M \to E$ is

$$\phi = \sum_i \phi^i_\alpha \, \hat{\alpha}_i$$

for $\phi_\alpha : U \to \mathbb{R}^{\mathrm{rk}(E)}$

Then because

$$\nabla(\hat{\alpha}_i) \in \Omega^1(U_\alpha, E) = \Omega^1(E) \otimes \Gamma_U E$$

we may write

	using $\phi = \sum_i \phi^i_\alpha \, \hat{\alpha}_i$	**using** $\phi_\alpha : U \to \mathbb{R}^{\mathrm{rk}(E)}$
	$\nabla(\hat{\alpha}_i) = \sum_j A^j_j \otimes \hat{\alpha}_j$	
		$\nabla(\phi_\alpha) = \mathrm{d}\phi_\alpha + A_\alpha \phi_\alpha$ where $A_\alpha \in \Omega^1(U, \mathfrak{gl}(n, \mathbb{R}))$
		$\nabla = \mathrm{d} + A_\alpha$

induced connections

induced from	**induced on**	**looks like**	
(E, ∇)	(E^*, ∇^*)		
	$E \otimes F$		
	$E \oplus F$		
	$E \times_\rho V$		

exterior derivative of vector bundle-valued forms

$$\mathrm{d}_\nabla : \Omega^k(M, E) \to \Omega^{k+1}(M, E)$$

such that

$$\mathrm{d}_\nabla(\omega \otimes s) = \mathrm{d}\omega \otimes s + (-1)^k \omega \wedge \nabla\phi$$

curvature

Definition. Curvature of a connection on a vector bundle

The **curvature** of the connection ∇ on $E \to M$ is a 2-form

$$F_\nabla \in \Omega^2(M, \mathrm{End}(E))$$

such that

$$F_\nabla(X, Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$

$$(\mathrm{d}_\nabla)^2 \sigma = F_\nabla \wedge \sigma$$

Current note has 0 direct children and 0 total descendants.

- structures based on sets
  - manifolds
    - smooth $\mathbb{R}$-manifold
      - Vector bundle on smooth manifolds
        - Connection on a smooth vector bundle on a smooth manifold

And it has 5 siblings.

- structures based on sets
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