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# Connection on a smooth vector bundle on a smooth manifold

Definition. Connection on a smooth vector bundle on a smooth manifold

Let  $E \rightarrow M$  be a smooth  $\mathbb{R}$ -vector bundle over a smooth manifold  $M$  with the space of sections  $\Gamma E$ . A **connection** on  $E \rightarrow M$  is a  $\mathbb{R}$ -linear map

$$\nabla : \Gamma E \rightarrow \Gamma(T^*M \otimes E)$$

such that

$$\nabla(f\phi) = \mathrm{d}f \otimes \phi + f\nabla(\phi)$$

for all  $f \in \mathcal{C}^\infty(M), \phi \in \Gamma E$ .

$$\nabla : \Omega^0(M, E) \rightarrow \Omega^1(M, E)$$

## locally

On a trivialization  $U_\alpha$  of  $E \rightarrow M$ , the section  $\phi : M \rightarrow E$  is

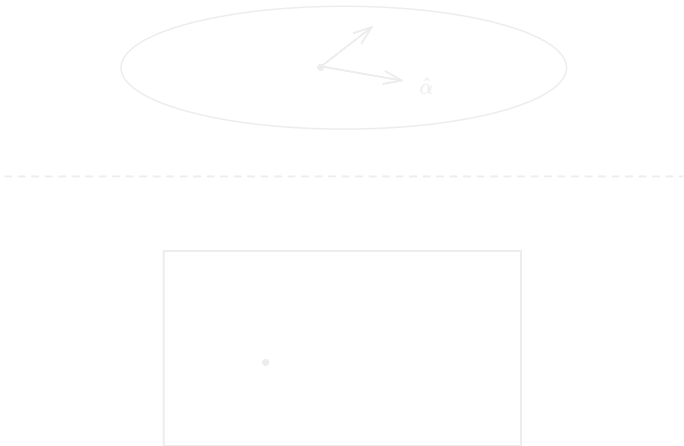
$$\phi = \sum_i \phi_\alpha^i \hat{\alpha}_i$$

for  $\phi_\alpha : U \rightarrow \mathbb{R}^{\mathrm{rk}(E)}$

Then because

$$\nabla(\hat{\alpha}_i) \in \Omega^1(U_\alpha, E) = \Omega^1(E) \otimes \Gamma_U E$$

we may write



|  |                                                                |                                                                                                                                        |
|--|----------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------|
|  | using $\phi = \sum_i \phi_\alpha^i \hat{\alpha}_i$             | using $\phi_\alpha : U \rightarrow \mathbb{R}^{\mathrm{rk}(E)}$                                                                        |
|  | $\nabla(\hat{\alpha}_i) = \sum_j A_j^i \otimes \hat{\alpha}_j$ |                                                                                                                                        |
|  |                                                                | $\nabla(\phi_\alpha) = \mathrm{d}\phi_\alpha + A_\alpha \phi_\alpha$<br>where $A_\alpha \in \Omega^1(U, \mathfrak{gl}(n, \mathbb{R}))$ |
|  |                                                                | $\nabla = \mathrm{d} + A_\alpha$                                                                                                       |

## induced connections

| induced from  | induced on        | looks like |  |
|---------------|-------------------|------------|--|
| $(E, \nabla)$ | $(E^*, \nabla^*)$ |            |  |
|               | $E \otimes F$     |            |  |
|               | $E \oplus F$      |            |  |
|               | $E \times_\rho V$ |            |  |
|               |                   |            |  |

## exterior derivative of vector bundle-valued forms

$$\mathrm{d}_\nabla : \Omega^k(M, E) \rightarrow \Omega^{k+1}(M, E)$$

such that

$$\mathrm{d}_\nabla(\omega \otimes s) = \mathrm{d}\omega \otimes s + (-1)^k \omega \wedge \nabla s$$

## curvature

Definition. Curvature of a connection on a vector bundle

The **curvature** of the connection  $\nabla$  on  $E \rightarrow M$  is a 2-form

$$F_\nabla \in \Omega^2(M, \mathrm{End}(E))$$

such that

$$F_\nabla(X, Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]}$$

| 
$$(\mathrm{d}_\nabla)^2 \sigma = F_\nabla \wedge \sigma$$

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