Info

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Modular forms

Definition. Weakly modular forms of *level* $SL_{\mathbb{Z}}(2)$ on upper-half plane

Let $k \in \mathbb{N}$. A meromorphic function

$$f : H \to \mathbb{C}$$

is **weakly modular** of weight k if

$$\tau \in \mathbb{H}, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_{\mathbb{Z}}(2) \implies f(\begin{bmatrix} a & b \\ c & d \end{bmatrix}(\tau)) = (c\tau + d)^k f(\tau)$$

- If this rule is satisfied for $S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$ then it holds for all $SL_{\mathbb{Z}}(2)$. This means f is weakly modular of weight k if

$$f(\tau + 1) = f(\tau) \text{ and } f\left(-\frac{1}{\tau}\right) = \tau^k f(\tau)$$

  - Weakly modular functions are $\mathbb{Z}\{1\}$-periodic.
- Weakly modular of **weight** 0 is simply $SL_{\mathbb{Z}}(2)$-invariance of f.
- Weakly modular of **weight** 2 is simply $SL_{\mathbb{Z}}(2)$-invariance of

$$f(\tau)\mathrm{d}\tau$$

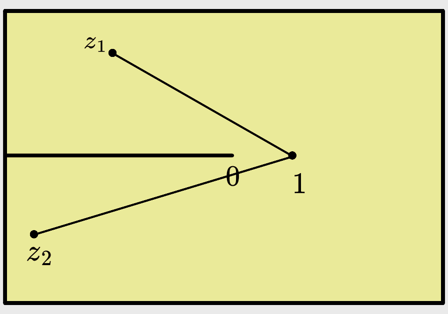
modular forms

- $$\begin{aligned} H &\to D \setminus \{0\} \\ \tau &\mapsto e^{2\pi i \tau} \end{aligned}$$

  is a holomorphic $\mathbb{Z}$-periodic map
  - $$\Im\tau \to \infty \iff q = e^{2\pi i \tau} \to 0$$
  - $$\Im\tau \to 0 \iff |q| \to 1$$
  - $\partial H = \mathbb{R}$ "goes to" $\partial D = S^1$
- We have the inverse

  Definition. The complex logarithm

  $$\begin{aligned} \mathbb{C} \setminus (-\infty, 0] &\to \mathbb{C} \\ \log(z) &:= \int_{1 \mapsto z} \frac{\mathrm{d}w}{w} \end{aligned}$$

  on

  $$D \setminus \mathbb{R}_{\leq 0} \to H$$

- If

  $$f : H \to \mathbb{C}$$

  is holomorphic/meromorphic such that $f(z+1) = f(z)$ ($\mathbb{Z}$-periodic)

  $$\iff$$

  there is a holomorphic/meromorphic $g : \mathbb{D}^* \to \mathbb{C}$ such that

  $$f(\tau) = g(e^{2\pi i \tau})$$

  given by the composition of log

  $$g = f \circ \frac{\log}{2\pi i}$$

  which is well-defined as
  - two different branches of log differ by $2\pi i k$ then

  $$f\left(\frac{\log q + 2\pi i k}{2\pi i}\right) = f\left(\frac{\log q}{2\pi i}\right)$$

  match
- Now g has a Laurent series

  $$g = \sum_{n \in \mathbb{Z}} a_n q^n \text{ on } D \setminus \{0\}$$

  - which gives f has a Fourier expansion

  $$f(\tau) = \sum_{n \in \mathbb{Z}} a_n e^{(2\pi i) n \tau}$$

Definition. Holomorphicity at ∞ of $\mathbb{Z}$-periodic functions on H

The holomorphic

$$f : H \to \mathbb{C}$$

is **holomorphic at** ∞ if the corresponding

$$g : D \setminus \{0\} \to \mathbb{C}$$

extends to $0 \in D$. This is equivalent to existence of

$$\lim_{\mathrm{im}(\tau)\to\infty} f(\tau)$$

or even just f being bounded as $\mathrm{im}(\tau) \to \infty$.

Definition. Modular forms of *level* $SL_{\mathbb{Z}}(2)$ on upper-half plane

A weakly modular function

> **Definition.** Weakly modular forms of *level* $SL_{\mathbb{Z}}(2)$ on upper-half plane
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> is **weakly modular** of weight k if
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> $$\tau \in \mathbb{H}, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_{\mathbb{Z}}(2) \implies f(\begin{bmatrix} a & b \\ c & d \end{bmatrix}(\tau)) = (c\tau + d)^k f(\tau)$$

is a **modular form of weight** k if f is holomorphic on H and at ∞.
The $\mathbb{C}$-vector space of modular forms of wight k is denoted by

$$\mathrm{Modular}_k(SL_{\mathbb{Z}}(2))$$

and

$$\mathrm{Modular}(SL_{\mathbb{Z}}(2)) := \bigoplus_{k\in\mathbb{Z}} \mathrm{Modular}_k(SL_{\mathbb{Z}}(2))$$

forms a $\mathbb{Z}$-graded $\mathbb{C}$-algebra.

Current note has 0 direct children and 0 total descendants.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $_\Gamma\backslash^{\mathbb{R}^n}, {}_\Gamma\backslash^{\mathbb{C}^n}$
    - Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
      - Modular forms

And it has 19 siblings.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $_\Gamma\backslash^{\mathbb{R}^n}, {}_\Gamma\backslash^{\mathbb{C}^n}$
    - Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
      - Approximation of holomorphic functions on $\mathbb{C}$ by rational functions
      - Embedding unit disk D into $\mathbb{C}^3$
      - Factorization of holomorphic functions on $\mathbb{C}$
      - Global holomorphic functions
      - Equivalent descriptions of holomorphic functions
      - Meromorphic functions on the complex plane
      - Holomorphic functions meromorphic at infinity
      - Modular forms
      - Reconstructing meromorphic functions from stalks
      - Extending holomorphic functions by reflections
      - Rotational symmetrization of holomorphic functions
      - Sheaf of holomorphic functions on $\mathbb{C}$
      - $\mathscr{O}(U)$
      - $\mathscr{O}(\mathbb{C})$
      - $\mathscr{O}(D)$
      - $\mathscr{O}^p(H^2_{\mathrm{U}})$
      - $\mathscr{O} \cap L^p$
      - $\mathscr{O}(S^1)$
      - Zeros and singularities of holomorphic functions