Info

This note, that was found here, as a part of a collection
is written (completely with human hands) by Rupadarshi Ray,
was created on November 30, 2025 9:59:08 PM,
was last modified on January 24, 2026 7:06:48 PM, and
was exported on September 7, 2026 3:10:04 PM.

T-closed, exp-closed closed subgroups of $GL(n,\mathbb{R}) \leftrightarrow$ completely geodesic submanifolds of $\mathsf{Pd}(n,\mathbb{R})$

Definition. Embedded *completely* geodesic submanifolds

Let $S \subseteq M$ be a connected, *embedded* submanifold of a uniquely geodesic Riemannian manifold (M, g). Then S is called **completely geodesic** if for every g-geodesic $\gamma : \mathbb{R} \to M$, if γ meets S in two points then this implies $\gamma(\mathbb{R}) \subset \iota(S)$.

> **Proposition:** Let $P \subseteq \mathsf{Pd}(n,\mathbb{R})$ be a completely geodesic, closed and embedded submanifold containing $I \in P$. Then P is also a CAT(0) symmetric space.

Let $G \leq GL(n,\mathbb{R})$ be a closed, T-closed and exp-closed subgroup.

$$K := G \cap O(n,\mathbb{R})$$
$$P := G \cap \mathsf{Pd}(n,\mathbb{R}) \ni I$$
$$\mathfrak{p} := T_I G \cap \mathsf{Sym}(n,\mathbb{R})$$

Then

$$\boldsymbol{t}_G(I) = P = \exp_I(\mathfrak{p}) \cong_G {}^{G}/_{K}$$

is a **completely geodesic, closed and embedded submanifold**, thus also a CAT(0) symmetric space. Moreover, $K \leq G$ is a maximal compact subgroup which is unique upto conjugacy and the map

$$K \times P \to G$$
$$(k, m) \mapsto km$$

is a diffeomorphism.

Let $V \subseteq \mathsf{Pd}(n,\mathbb{R})$ is a completely geodesic, embedded, closed submanifold containing $I \in V$ then

$$G := \{g \in GL(n,\mathbb{R}) | \boldsymbol{t}_g(V) = V\} \leq GL(n,\mathbb{R})$$

is a closed, exp-closed, T-closed subgroup such that

$$V = G \cap \mathsf{Pd}(n,\mathbb{R})$$

$\boldsymbol{t}_G(I) = P = \exp_I(\mathfrak{p})$

As

$$g \in G \implies g^{\mathsf{T}} \in G \implies gg^{\mathsf{T}} \in G$$
$$gg^{\mathsf{T}} = \boldsymbol{t}_g(I) \in \mathsf{Pd}(n,\mathbb{R})$$

we have the restricted orbit map to $G \leq GL(n,\mathbb{R})$

$$G \to P := G \cap \mathsf{Pd}(n,\mathbb{R})$$
$$g \mapsto \boldsymbol{t}_g(I) = gg^{\mathsf{T}}$$

- If $X \in \mathsf{Sym}(n,\mathbb{R})$ and $p = \exp(X) \in G$ then $\exp\left(\frac{1}{2}X\right) \in G$ by assumption. Then

  $$\boldsymbol{t}_{\exp(\frac{1}{2}X)}(I) = \exp\left(\frac{1}{2}X\right)^2 = p$$

  Thus we have $P \subseteq \boldsymbol{t}_G(I)$ implying

  $$P := G \cap \mathsf{Pd}(n,\mathbb{R}) = \boldsymbol{t}_G(I)$$

- The **stabilizer** of I in G

  $$\{g \in G | \forall p \in M, gIg^{\mathsf{T}} = I\} = \underbrace{O(n,\mathbb{R})}_{\text{compact}} \cap G =: K$$

  which is compact.

- Now by exp-closedness

$$\log(\underbrace{G \cap \mathsf{Pd}(n,\mathbb{R})}_{M}) \subset T_I G$$

and also

$$\exp(\underbrace{T_I G \cap \mathsf{Sym}(n,\mathbb{R})}_{\mathfrak{p}}) \subset \underbrace{G \cap \mathsf{Pd}(n,\mathbb{R})}_{P}$$

- Thus

$$P = \exp(\underbrace{T_I G \cap \mathsf{Sym}(n,\mathbb{R})}_{=:\mathfrak{p}})$$

  - Hence, P is a closed and embedded submanifold of $\mathsf{Pd}(n,\mathbb{R})$ as it is a diffeomorphic image of an closed and embedded submanifold.

P is G-diffeomorphic to ${}^{G}/_{K}$

> **Proposition:** The action
>
> $$\boldsymbol{t} : GL(n,\mathbb{R}) \curvearrowright \mathsf{Pd}(n,\mathbb{R})$$
>
> is **proper**.

Let $\{p_i\} \subset \mathsf{Pd}(n, \mathbb{R})$ and $\{g_i\} \subset GL(n, \mathbb{R})$ be sequences such that

$$p_i, \boldsymbol{t}_{g_i}(p_i) = g_i p_i g_i^\mathsf{T}$$

converge.

- As $G \leq GL(n, \mathbb{R})$ is a closed subgroup, then by

  > Let $G \curvearrowright X$ is a proper action and $H \leq G$ is a closed subgroup. Then
  >
  > $$H \curvearrowright X := H \hookrightarrow G \curvearrowright X$$
  >
  > is a **proper action**.

  action $G \curvearrowright \mathsf{Pd}(n, \mathbb{R})$ is also **proper**.

  - Thus by

    > Let $\theta : G \curvearrowright M$ be a smooth left Lie group action on a smooth manifold M and $p \in M$. Then the orbit map
    >
    > $$\begin{aligned} \theta_{(p)} : G &\to M \\ g &\mapsto gp \end{aligned}$$
    >
    > is smooth and has constant rank, so the isotopy group $G_p = \theta_{(p)}^{-1}(p)$ is a **closed subgroup** of G. Then
    >
    > $$\theta_{(p)} : \frac{G}{G_p} \hookrightarrow M$$
    >
    > is an injective immersion making the image, that is, the orbit $\theta^G(p)$ is an **immersed submanifold**
    >
    > $$\iota : (G\{p\}, \theta_{(p)*}\mathfrak{T}_{G/G_p}, \theta_{(p)*}\mathscr{A}_{G/G_p}) \hookrightarrow M$$
    >
    > Moreover, if the action θ is *proper* then the orbit map $\theta_{(p)}$ is a proper smooth embedding (so in particular the orbit $\theta^G(p)$ is closed/properly embedded) and each stabilizer G_p is compact.

    the orbit

    $$M := \boldsymbol{t}_G(I)$$

    is an **embedded submanifold** of $\mathsf{Pd}(n, \mathbb{R})$.

  - Thus in particular, we have a G-equivariant diffeomorphism

    $$G/_K \cong_{\mathsf{Man}_G} P$$

    by the orbit map.

Current note has 0 direct children and 0 total descendants.

- stem
  - $GL(n)(\mathbb{R})$
    - T-closed, exp-closed closed subgroups of $GL(n, \mathbb{R})$ $\leftrightarrow$ completely geodesic submanifolds of $\mathsf{Pd}(n, \mathbb{R})$

And it has 4 siblings.

- stem
  - $GL(n)(\mathbb{R})$
    - T-closed, closed subgroups of $GL(n, \mathbb{R})$
    - Subgroups of $GL(n, \mathbb{R})$
    - Compact subgroups of $GL(n)(\mathbb{R})$
    - T-closed, exp-closed closed subgroups of $GL(n, \mathbb{R})$ $\leftrightarrow$ completely geodesic submanifolds of $\mathsf{Pd}(n, \mathbb{R})$