Info

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Differential forms on smooth manifolds

Shouldn't *differential forms* be called *differential fluxes*?

Definition. Differential forms on smooth manifolds

$$(\Omega^\bullet(M), +, \ , \wedge)$$

Differential forms are the elements of the alternating $\mathbb{R}$-algebra also $\mathcal{C}^\infty(M)$-algebra

$$(\Omega^\bullet(M), +, \wedge)$$

which are equivalently

- $$\boldsymbol{\Gamma}\boldsymbol{\Lambda}^\bullet T^*(M) := \bigoplus_k \boldsymbol{\Gamma}\boldsymbol{\Lambda}^k T^*(M)$$

  and define the wedge product of two sections *pointwise*

  $$(\omega \wedge \eta)_p := \omega_p \wedge \eta_p$$

- or

$$\mathrm{Alt}_{\mathcal{C}^\infty(M)}((\boldsymbol{\Gamma} TM)^k, \mathcal{C}^\infty(M))$$

Locally

$$\omega = \sum_I f_I \mathrm{d}x_{I_1} \wedge \mathrm{d}x_{I_2} \dots$$

or globally

$$\omega(X_1, X_2 \dots)$$

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[1]

[2]

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	exterior covectors	space of exterior covectors $\boldsymbol{\Lambda}^k V^*$	differential forms	space of differential k-forms $\Omega^k(M)$	on the algebra of differential forms	on a local chart
Interior multiplication	$(\iota_v\omega)(-) :=$	$\iota_v : \boldsymbol{\Lambda}^k V^* \to$ such that		$\iota_X : \Omega^k(M)$		

	exterior covectors	space of exterior covectors $\Lambda^k V^*$	differential forms	space of differential k-forms $\Omega^k(M)$	on the algebra of differential forms	on a local chart
Exterior derivative	-	-				
Pullback						
Lie derivative	-	-				
pullback, Lie derivative and exterior derivative						

- $\Omega^k(M)$
  - $\mathbb{R}$-vector space
  - $\mathcal{C}^\infty(M)$-module
  - $\int$ on smooth chains
  - pullback and functor
  - sheaf of vector space/module
- $\Omega(M)$
  - d
  - alternating $\mathbb{R}$-algebra
    - functor $\mathsf{Man} \to \mathsf{Alg}_\mathbb{R}$
    - d as a natural transformation
  - alternating $\mathcal{C}^\infty(M)$-algebra
  - $(\Omega(M), \mathrm{d})$ chain complex
  - $(\Omega(M), \wedge, \mathrm{d})$ alternating algebra+chain complex
  - sheaf of algebra
    - d as morphism of sheaves
- $(\bigoplus_k \Omega^{2k}(M)) \oplus (\bigoplus_k \Omega^{2k+1}(M))$
  - $\mathbb{Z}_2$-graded $\mathbb{R}$-algebra
- Lattice of forms with integer integral $\Omega^k_\mathbb{Z}(M)$
  - $H^k(M, \mathbb{Z}) \to \Omega^k(M, \mathbb{Z})$
  - "torus"

$$\frac{\Omega^k(M)}{\Omega^k_\mathbb{Z}(M)}$$

- $\Omega^k_c(M)$
  - $\int$ on k-submanifolds
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	$f : S \to M$	cohomology	of submanifolds $\iota_N : N \leq M$	sheaf
relative to a smooth map	Definition. D f r a r Let $f : S \to M$ be a smooth map between smooth manifolds. Then $(\Omega^\bullet[f], \mathrm{d})$ $\Omega^\bullet[f] := \bigoplus$ $\Omega^k[f] := \Omega^k($ $\mathrm{d}(\omega, \theta) := (\mathrm{d}\omega$ is the **cocomplex of differential forms relative to the map f.**	consists of closed forms on M that become exact when pulled back by f	Chain complex of differential forms relative to a submanifold	
kernel	$f : S \to M$ $\ker f^*$		subalgebra+sub complex of forms which is 0 when restricted to a submanifold which is the kernel of $\Omega^\bullet(M; N) := \ker$	
cokernel	$\Omega^\bullet(f : S/M) :=$		$\Omega^\bullet(N/M) := \frac{\Omega}{\iota_N^*}$	

	$f : S \to M$	cohomology	of submanifolds $\iota_N : N \leq M$	sheaf

- $$\mathfrak{L}_X = \mathrm{d}\iota_X + \iota_X \mathrm{d}$$
    - $\mathrm{d}(X\phi) = \mathfrak{L}_X \mathrm{d}\phi$
- Vector space valued differential forms
- Lie algebra valued differential forms

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integrating differential forms on manifolds

Definition.

$$\int_M \omega := \int_{\phi(U)} (\phi^{-1})^* \omega$$

exterior derivative

For every smooth manifold M, $\exists!$ linear maps

$$\mathrm{d}^k : \mathrm{Forms}^k(M) \to \mathrm{Forms}^{k+1}(M)$$

for each $k \geq 0$ satisfying

$$\begin{aligned} \mathrm{d}^0 f(X) &= Xf \\ \mathrm{d}^{k+l}(\omega \wedge \eta) &= \mathrm{d}^k \omega \wedge \eta + (-1)^k \omega \wedge \mathrm{d}^l \eta \\ \mathrm{d}^{k+1} \mathrm{d}^k &= 0 \end{aligned}$$

exact and closed

Definition. Closed k-forms on M are $\ker(\mathrm{d}^k)$ and exact k-forms are $\mathrm{im}\ (\mathrm{d}^{k-1})$

de Rham cohomology
Differential 1-forms on smooth manifolds

cohomologous forms

Definition. $\omega_1, \omega_2 \in \Gamma\Lambda^k(T^*M)$ are *cohomologous* if $\omega_1 - \omega_2 \in \mathrm{im}\ (\mathrm{d}^{k-1})$ or exact

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Current note has 7 direct children and 14 total descendants.

- structures based on sets
    - manifolds
        - smooth $\mathbb{R}$-manifold
            - Tangents on a smooth manifold
                - Differential forms on smooth manifolds
                    - Differential 1-forms on smooth manifolds
                        - Closed differential 1-forms on smooth manifolds
                        - When are closed differential 1-forms exact on a smooth manifold?
                        - Conservative 1-forms
                    - Chain complex of differential forms on a smooth manifold
                        - Homotopy invariance of cohomology of differential forms
                    - Exterior derivative operator on space of smooth differential forms
                    - Functoriality of Forms^k on smooth manifolds
                    - The integral map on space of differential forms
                    - Three different relative chain complexes of differential forms
                        - Differential forms relative to a smooth map
                        - Chain complex of differential forms relative to a submanifold
                    - The sheaf of differential forms on open subsets of manifolds
                        - Isomorphism of the de Rham cohomology with the sheaf cohomology of locally $\mathbb{R}$-constant functions

And it has 4 siblings.

- structures based on sets
    - manifolds
        - smooth $\mathbb{R}$-manifold
            - Tangents on a smooth manifold
                - Cotangent space on a smooth manifold T_p^*M
                - Connection on the tangent bundle of a smooth manifold
                - Differential forms on smooth manifolds
                - Tangent space of a smooth manifold

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1. On the Visualization of Differential Forms (Finished except for appendix) – Random Math Stuff (ulisboa.pt) ↩
2. piponi-on.pdf (yaroslavvb.com) ↩