Info

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Simple singularities of holomorphic functions

AKA Du Val singularity, rational double point also called *simple surface singularity*, *Kleinian singularity*.

$$\begin{tikzcd} \mathbb{C}^2 \\ (z_1, z_2) \arrow[d] \arrow[r] & V(R) \subseteq \mathbb{C}^3 \\ & (X, Y, Z)(z_1, z_2) \\ \mathbb{C}^2/\Gamma \arrow[ur, "\cong"'] \end{tikzcd}$$

thus we have

$$\mathbb{C}^2/\Gamma \cong V(R)$$

modality $= 0$	**normal form**	**simply laced Dynkin diagram**	**monodromy group** $\cong$ **subgroup** Γ^* **of** $SU(2)$ **that is the preimage of finite subgroup** $\Gamma \leq SO(3)$	$R(X, Y, Z)$	$V(R) \cong \mathbb{C}^3/\Gamma^*$ **which is isomorphic to the level bifurcation set**
$A_k, k \geq 1$	x^{k+1}		$\mathbb{Z}_{k+1}$	$X^{k+1} + YZ$	
$D_k, k \geq 4$	$x^2 y + y^{k+1}$		binary dihedral $D_{2(k-2)}$	$XY^2 - X^{k-1} +$	
E_6	$x^3 + y^4$		binary tetrahedral		
E_7	$x^3 + xy^4$		binary oct		
E_8	$x^3 + y^5$		binary iso		

[1]

A_n

A_n **for** $n =$	**more examples of functions equivalent to** f	**normal form** f	**miniversal deformation** $+g_\lambda$	$V(f + g_\lambda)$	$\Delta = 0$	$V(\Delta)$	$R(X, Y, Z)$	$V(R)$
nope!		x is not a singularity?						
1	$x^2 - y^2 -$, $x^2 + y^2 +$	x^2	$+\lambda$		$\lambda = 0?$	fold?	$X^2 + YZ$	
2	$x^3 - y^2 -$, $x^3 + y^2 -$	x^3	$+\lambda_1 x + \lambda$		$27\lambda_0^2 + 4\lambda$	cusp?	$X^3 + YZ$	
3		x^4	$+\lambda_2 x^2 +$		16 $-128\lambda_2^2\lambda$ -27λ	swallowtail?		
4		x^5				butterfly?		
5		x^6						
n		x^{n+1}	$+\sum_{i=0}^{n-1} \lambda$		discriminant of the monic polynomial of degree $n+1$			

[2]

Example: The subgroup $C_n \subset SL(2, \mathbb{C})$ is given by the matrices $\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$ where ζ runs through all n-th roots of unity. Let us denote coordinates on $\mathbb{C}^2$ by U and V. Then one may take

$$X = UV, \quad Y = U^n, \quad \text{and} \quad Z = -V^n$$

as fundamental invariants satisfying the relation $X^n + YZ = 0$.

$$\left\langle \begin{bmatrix} \exp\left(i\frac{\pi}{n}\right) & 0 \\ 0 & \exp(-i\frac{\pi}{n}) \end{bmatrix} \right\rangle \hookrightarrow SU(2) \curvearrowright \mathbb{C}^2$$

The invariant polynomials in $\mathbb{C}[e_1, e_2]$

$$X := e_1 e_2$$
$$Y := e_1^n$$
$$Z := -e_2^n$$

which satisfy

$$X^n + YZ = 0$$

> Claim: variety/manifold? of non-regular orbits in $\mathbb{C}^2/\Gamma \cong$ level bifurcation set $\Delta = 0$

We look at the homeomorphism

$$\mathbb{C}^n/S_n \cong_{\mathsf{Top}} X^n + \mathbb{C}[X]_{n-1}$$
$$\prod_{i=1}^{n}(X-\alpha_i) \mapsto (\lambda_i) \text{ in } X^n + \sum_{i=0}^{n-1}\lambda^i X^i$$

and try to construct something similar for $\Gamma = \mathbb{Z}_n$

$$\begin{bmatrix}\exp\left(i\frac{\pi}{n}\right) & 0 \\ 0 & \exp\left(-i\frac{\pi}{n}\right)\end{bmatrix}\begin{bmatrix}z_1 \\ z_2\end{bmatrix} = \exp\left(i\frac{\pi}{n}\right)\begin{bmatrix}z_1 \\ 0\end{bmatrix} + \exp\left(-i\frac{\pi}{n}\right)\begin{bmatrix}0 \\ z_2\end{bmatrix}$$

- Regular orbits $\leftrightarrow$ ones with $|\Gamma|$ number of elements
- Non-regular orbits $\leftrightarrow$ $z_i = 0$ or $z_1 = \pm z_2$ $\leftrightarrow$ $(X)(X-z_2)$ or $(X)(X-z_1)$ or $(X-z_1)^2$ or $X^2 - z_1^2$
- ???

A_1

Consider the germ of A_1-singularity $S = V(x^2+y^2+z^2) \subset \mathbf{A}^3$. Consider the blow-up of this singularity.

$$\tilde{\mathbf{A}}^3 = \{((x,y,z),(u:v:w)) \in \mathbf{A}^3 \times \mathbf{P}^2 | xv = yu, xw = zu, yw = zu\}.$$

Take the chart $u \neq 0$, i.e. $u = 1$. We get

$$\begin{cases} x &= x \\ y &= xv \\ z &= xw \end{cases}$$

What is $\tilde{S} = \overline{\pi^{-1}(S \backslash \{0\})}$? Consider first $x \neq 0$ (it means that we are looking for $\pi^{-1}(S\backslash\{0\})$).

$$x^2 + x^2v^2 + x^2w^2 = 0, x \neq 0,$$

or

$$1 + v^2 + w^2 = 0, x \neq 0.$$

In order to get $\tilde{S}$ we should allow x to be arbitrary. In this chart $\tilde{S}$ is a cylinder $V(1+v^2+w^2) \subset \mathbf{A}^3$. What is $\pi^{-1}(0)$? Obviously it is the intersection of $\tilde{S}$ with the exceptional plane $((0,0,0),(u:v:w))$. In this chart we just have to set $x = 0$ in addition to the equation of the surface $\tilde{S}$.

$$\pi^{-1}(0) = \begin{cases} 1+v^2+w^2 &= 0 \\ x &= 0. \end{cases}$$

[3]

A_2

✓ https://www.desmos.com/3d/mhd4xpgf9h $V(R)$

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frame: PDF
style: height: 900px;
urlSuffix: www.desmos.com/3d/mhd4xpgf9h
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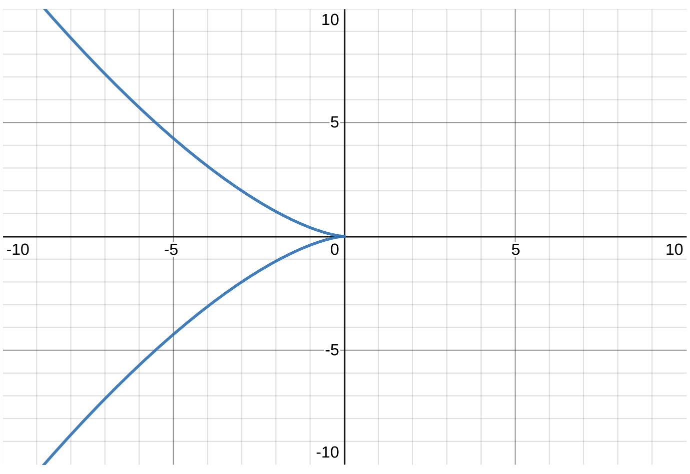
the zero-level-set of a type A_2-*singularity*

Level bifurcation set: $27\lambda_0^2 + 4\lambda_1^3 = 0$



Current note has 0 direct children and 0 total descendants.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients ${}_\Gamma\backslash^{\mathbb{R}^n}, {}_\Gamma\backslash^{\mathbb{C}^n}$
    - Ramified germs of smooth and holomorphic functions
      - Simple singularities of holomorphic functions

And it has 1 siblings.

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1. The ADE classification of singularities (singsurf.org) ↩
2. SingSurf: Cubic surface Visualizer ↩

3. mi.uni-koeln.de/~burban/singul.pdf ↩