ⓘ Info

This note, that was found here, as a part of a collection
is written (completely with human hands) by Rupadarshi Ray,
was created on June 23, 2026 4:38:30 AM,
was last modified on September 4, 2026 2:06:22 AM, and
was exported on September 7, 2026 3:10:03 PM.

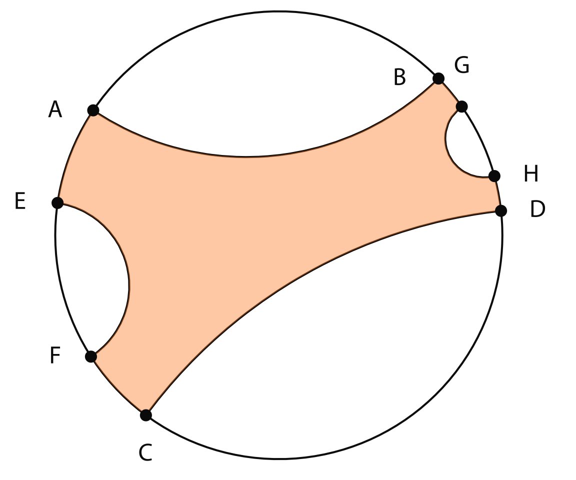
Classical Schottky subgroups of $O^+(1,n)(\mathbb{R})$

Definition. Classical Schottky subgroups of $\mathrm{Isom}(\mathbb{R}\boldsymbol{H}^n) \cong O^+(1,n)$

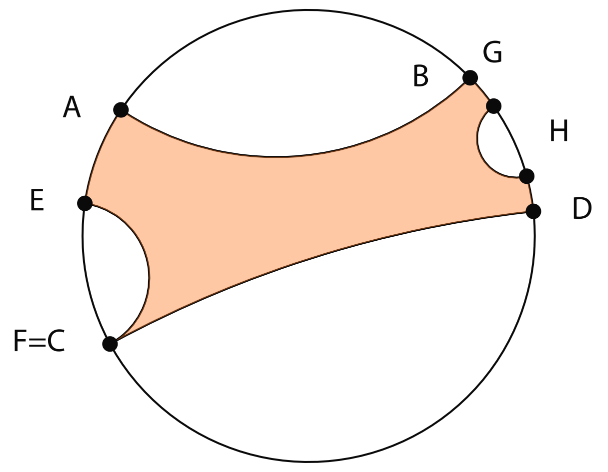
Consider a convex polyhedron P in $\mathbb{R}\boldsymbol{H}^n$ with even number of sides, and each side a hyperplane of $\mathbb{R}\boldsymbol{H}^n$ and a $\mathrm{Isom}(\mathbb{R}\boldsymbol{H}^n)$-side pairing Φ for P such that no side of P is paired to itself. The group generated by the side pairings

$$\Gamma := \langle \Phi \rangle$$

is called a **classical Schottky subgroup** of $\mathrm{Isom}(\mathbb{R}\boldsymbol{H}^n)$ for P.
If *no* two sides of P meet at infinity, then Γ is a **classical ultra-Schottky subgroup** *with respect to P*, otherwise it is a **classical sub-Schottky subgroup** *with respect to P*.



ultra



sub

Fundamental domain of each side-pairing map

Consider $2m$ disjoint hyperplanes $S_1, \dots, S_m, S'_1, \dots, S'_m \subseteq \mathbb{R}\boldsymbol{H}^n$ and $g_1, \dots g_m \in \mathrm{Isom}(\mathbb{R}\boldsymbol{H}^n)$ such that

- for each hyperplane $S_k^\bullet$, all the other hyperplanes lie on the same half-plane in $\mathbb{R}\boldsymbol{H}^n \setminus S_k^\bullet$
- for each k

  $$g_k(S_k) = S'_k$$
  $$P_k \cap g_{S_k}(P_k) = S'_k$$

  where P_k is the convex polyhedra with sides S_k, S'_k
- $P := \bigcap_k P_k$ has boundary

  $$\partial P = \bigsqcup_k (S_k \sqcup S'_k)$$

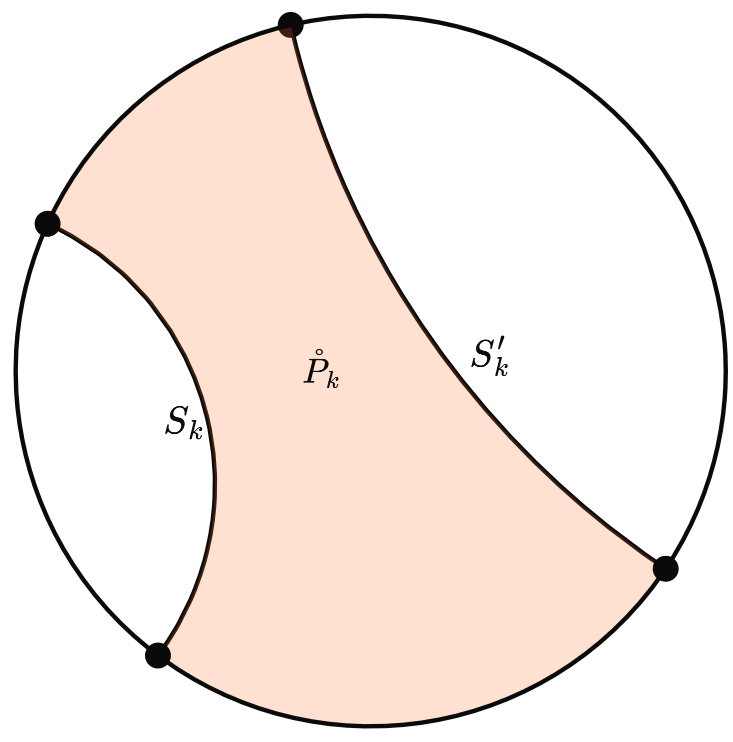
- $$\mathring{P}_k \cup \mathring{P}_j = \mathbb{R}\boldsymbol{H}^n$$

- $$D := \bigcap_k \mathring{P}_k \neq \emptyset$$

> **Proposition:** For all $N \neq 0$ we have
>
> $$\mathring{P}_k \cap g_k^N(\mathring{P}_k) = \emptyset$$
>
> for each k.
> This implies g_k is of infinite order for each k.

- The open set $\mathbb{R}\boldsymbol{H}^n \setminus (S_k \cup S'_k)$ has 3 connected components

  $$\mathbb{R}\boldsymbol{H}^n \setminus (S_k \cup S'_k) = \mathring{P}_k \sqcup$$

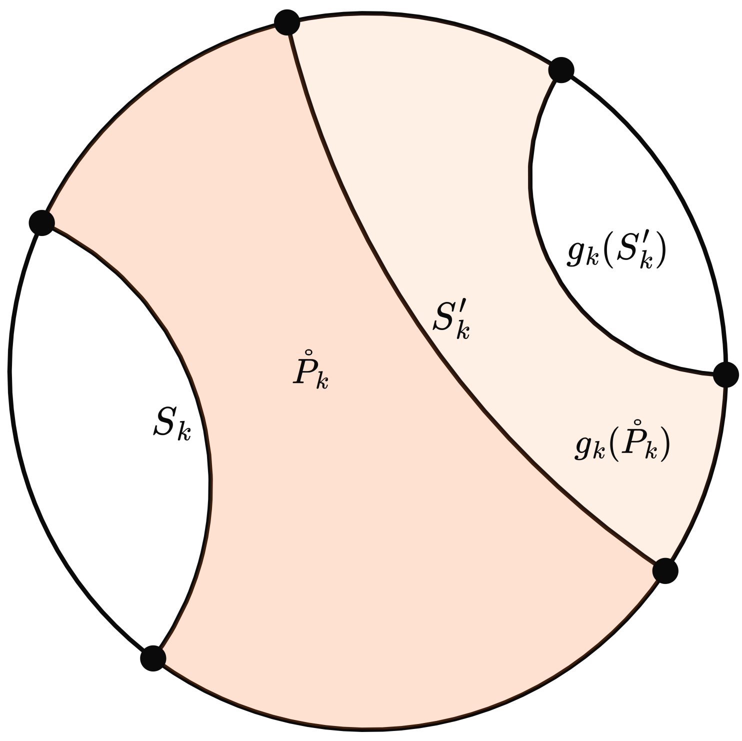
  

  -
- The connected set $g_k(\mathring{P}_k)$ must in one of the 3 components.
  - We know

    $$P_k \cap g_k(P_k) = S'_k$$

- This implies

$$g_k(\mathring{P}_k) \subseteq [S'_k]$$



-

- Similarly, $g_k^2(P_k)$ must be in one of the 3 components of $\mathbb{R}\boldsymbol{H}^n \setminus (S'_k \cup g_k(S'_k))$. Because

$$g_k(P_k) \cap g_k^2(P_k) = g_k(S'_k)$$

we conclude

$$g_k^2(\mathring{P}_k) \subseteq [g_k(S'_k)] \subseteq \mathbb{R}\boldsymbol{H}^n \setminus \mathring{P}_k$$

classical Schottky groups are free and discrete

> **Proposition:** Γ is freely generated by $g_1, \dots, g_m$.

Consider an element $g_{n_l}^{a_l} \dots g_{n_1}^{a_1} \in \Gamma$ for some $l > 0$.

- **Base case:** From

> **Proposition:** For all $N \neq 0$ we have
>
> $$\mathring{P}_k \cap g_k^N(\mathring{P}_k) = \emptyset$$
>
> for each k.
>
> This implies g_k is of infinite order for each k.

we have

$$g_{n_1}^{a_1}(D) \subseteq g_{n_1}^{a_1}(\mathring{P}_{n_1}) \subseteq \mathbb{R}\boldsymbol{H}^n \setminus \mathring{P}_{n_1}$$

- **Induction hypothesis:** For $k < l$ we assume

$$g_{n_k}^{a_k} \dots g_{n_1}^{a_1}(D) \subseteq \mathbb{R}\boldsymbol{H}^n \setminus \mathring{P}_{n_k}$$

- **Induction step:**

$$\begin{aligned} g_{k+1}^{a_{k+1}}(g_{n_k}^{a_k} \dots g_{n_1}^{a_1}(D)) &\subseteq g_{k+1}^{a_{k+1}}(\underbrace{\mathbb{R}\boldsymbol{H}^n \setminus \mathring{P}_{n_k}}_{\subseteq \mathring{P}_{n_{k+1}}}) \\ &\subseteq \mathbb{R}\boldsymbol{H}^n \setminus \mathring{P}_{n_{k+1}} \end{aligned}$$

- Therefore,

$$\begin{aligned} g_{n_l}^{a_l} \dots g_{n_1}^{a_1}(D) &\subseteq \mathbb{R}\boldsymbol{H}^n \setminus \mathring{P}_{n_l} \subseteq \mathbb{R}\boldsymbol{H}^n \setminus D \\ &\implies g_{n_l}^{a_l} \dots g_{n_1}^{a_1} \neq 1 \end{aligned}$$

- Therefore, Γ is freely generated by $g_1, \dots, g_m$.
- Because

$$D \cap g(D) = \emptyset$$

for each $g \in \Gamma \setminus \{1\}$, we conclude Γ is discrete.

A *classical Schottky subgroup* of $\mathrm{Isom}(\mathbb{R}\boldsymbol{H}^n)$ for a polyhedron P with $2m$ sides is a discrete, free subgroup of $\mathrm{Isom}(\mathbb{R}\boldsymbol{H}^n)$ of rank m, generated freely by the side pairings of P.

[1]

ultra

Let Γ be a *classical ultra-Schottky subgroup* of $\mathrm{Isom}(\mathbb{R}\boldsymbol{H}^n)$ for a Schottky polyhedron P. Then P is an exact, convex fundamendal polyhedron for Γ.

> A *classical Schottky subgroup* of $\mathrm{Isom}(\mathbb{R}\boldsymbol{H}^n)$ is finite sided and in particular **geometrically finite**?

An *classical ultra-Schottky subgroup* is finite sided and does not contain parabolic elements, and thus is **convex cocompact**.

limit set of classical ultra-Schottky groups

> **Proposition:**
>
> $$\partial_\infty P \subseteq \Omega(\Gamma)$$

> **Proposition:**

Definition. Let $P_1 := P$. Assuming P_i has been defined we choose P_{i+1} to tbe the union of P_i and the polyhedra in $\{\gamma P | \gamma \in \Gamma\}$ that share a side with P_i.

Then we have

$$\Lambda(\Gamma) \subseteq \bigcap_{i \geq 1} \overline{\mathbb{R}\boldsymbol{H}^n}^\infty \setminus \overline{P_i}^\infty$$

As

$$\bigcup_{\gamma \in \Gamma} \gamma P = \mathbb{R}\boldsymbol{H}^n$$

is an *exact tesselation* of $\mathbb{R}\boldsymbol{H}^n$, the family $\{\gamma P | \gamma \in \Gamma\}$ is *connected*.

Proposition: $\Lambda(\Gamma)$ is totally disconnected.

- Let $\xi, \eta \in \Lambda(\Gamma)$ be distinct limit points.
  - Choose k such that $\overline{\mathbb{R}\boldsymbol{H}^n}^\infty \setminus \overline{P_k}^\infty$ does not contain $\overline{\xi\eta}$.
  - Then by convexity η, ξ lies in different components of $\overline{\mathbb{R}\boldsymbol{H}^n}^\infty \setminus \overline{P_k}^\infty$.
  - Let U be the component of $\partial\mathbb{R}\boldsymbol{H}^n \setminus \overline{P_k}^\infty$ containing ξ and let V be the union of the remaining components.
  - Therefore

$$\Lambda(\Gamma) = \underbrace{U}_{\ni\xi} \sqcup \underbrace{V}_{\ni\eta}$$

  - Thus $\Lambda(\Gamma)$ is totally disconnected.

Current note has 0 direct children and 0 total descendants.

- stem
  - O+ R1 n
    - Discrete subgroups of $O^+(1,n)$ $\leftrightarrow$ hyperbolic n-manifolds/orbifolds
      - Schottky
        - Classical Schottky subgroups of $O^+(1,n)$
  - Objects stemming from $PSL(2)(\mathbb{C})$
    - Kleinian groups $\Gamma <_{\mathrm{d}} PSL(2)(\mathbb{C})$ $\leftrightarrow$ Hyperbolic 3-manifolds/orbifolds ${}_\Gamma\backslash^{\mathbb{R}\boldsymbol{H}^3}$
      - Kleinian ultra-Schottky groups $\leftrightarrow$ convex cocompact hyperbolic interior of 3-handlebodies

And it has 2 siblings.

- stem
  - O+ R1 n
    - Discrete subgroups of $O^+(1,n)$ $\leftrightarrow$ hyperbolic n-manifolds/orbifolds
      - Schottky
        - Classical Schottky subgroups of $O^+(1,n)$
        - Schottky subgroups of $O^+(1,n)$

---

1. John G. Ratcliffe - Foundations of Hyperbolic Manifolds (2019), p.610 ↩