Info

This note, that was found here, as a part of a collection
is written (completely with human hands) by Rupadarshi Ray,
was created on January 12, 2024 9:38:27 AM,
was last modified on August 29, 2026 6:12:04 PM, and
was exported on September 7, 2026 2:50:41 PM.

$PSL(2)(\mathbb{C}) \curvearrowright \mathbb{C}P^1 \cong_{\text{Man}} S^2$

#Wiki/group-action/Lie

Definition. Mobius transformations on $\mathbb{C}P^1$

The action $SL(2)(\mathbb{C})$ on $\mathbb{C}^2$ descends to the action of $PSL(2)(\mathbb{C})$ acts on $\mathbb{C}P^1$ by **Mobius transformations**

$$PSL(2)(\mathbb{C}) \curvearrowright \mathbb{C}P^1 \cong_{\text{Man}} S^2 \cong \mathbb{C} \cup \{\infty\}$$
$$z \xmapsto{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} \frac{az+b}{cz+d}$$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \mapsto (z \mapsto -z)$$

$GL(2)(\mathbb{C}) \curvearrowright \mathbb{C}^2$	$PGL(2)(\mathbb{C}) \cong PSL(2)(\mathbb{C}) \curvearrowright \mathbb{C}P^1 := \frac{\mathbb{C}^2 \setminus \{0}{\lambda v \sim v}$
eigenvector v	fixed point $\mathbb{C}\{v\}$
semi-simple/diagonalizable	has two fixed points

subgroup/element of $PSL(2)(\mathbb{C})$	**action on** $\mathbb{C}P^1 \cong S^2$
$\frac{\left\{\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}\right\}}{\{I, -I\}}$	translations on $\mathbb{C}$
$\frac{\left\{\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \middle\vert a \in \mathbb{C}^\times \right\}}{\{I, -I\}}$	dilations on $\mathbb{C}$ $z \mapsto az$
invertible upper triangular matrices modulo scaling $PGU(2, \mathbb{C}) := \frac{GU(2, \mathbb{C})}{\{\lambda I \vert \lambda \in \mathbb{C}^\times\}} \cong \frac{\left\{\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \middle\vert a \in \right.}{\{I, -}$	stabilizer of ∞ $\mathrm{AutMan}_{\mathbb{C}}(\mathbb{C})$ $z \mapsto az+b$
$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	inversion $z \mapsto \frac{1}{z}$
$PSU(2, \mathbb{C}) \cong SO(3, \mathbb{R})$	rotations of $S^2 \cong \mathbb{C}P^1$
$PSL(2)(\mathbb{R})$	stabilizer of $\mathbb{R}H^2, \mathbb{R}P^1$? and $\mathrm{AutMan}_{\mathbb{C}}(\mathbb{R}H^2)$
$PSO(2)(\mathbb{C})$ $\frac{\left\{\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \middle\vert a^2+b^2=1 \right\}}{\{I, -I\}}$	?
$N(\Lambda)$	$\mathrm{Aut}(\Lambda)$ for lattice $\Lambda \subseteq \mathbb{C}$

decompositions

- If $c = 0$,

$$\begin{bmatrix} \frac{a}{d} & \frac{b}{d} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{b}{d} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{a}{d} & 0 \\ 0 & 1 \end{bmatrix}$$

  any 2-rational transformation = translation ∘ dilation, unless one of them is identity.

- If $c \neq 0$,

fixed points

Fixed points of

$$z \mapsto \frac{az+b}{cz+d}$$

are roots of

$$cz + (d-a)z - b = 0$$

which is

- if $c = 0$, the fixed point is

$$z = \frac{b}{d-a}$$

- if $c \neq 0$
- if $a, b, c, d = 1$, the map is $\mathrm{Id}_{\mathbb{C}P^1}$

conjugacy types

	parabolic	elliptic	hyperbolic	loxodromic
$\mathrm{tr}^2 = \lambda + \dfrac{1}{\lambda} + 2$	$= 4$	$\in [0, 4)$	$\in (4, \infty)$	$\mathbb{C} \setminus [0, \infty)$
$\lambda, \dfrac{1}{\lambda} = \dfrac{-2 + \mathrm{tr}^2}{}$	$\lambda = 1$	$\lambda \in U(1) \setminus \{1\}$	$\lambda > 0$	$\lambda \in \mathbb{C} \setminus U(1) \setminus [0,$
	translation $z \mapsto z + 1$	rotation $z \mapsto e^{i\theta} z$	scaling $z \mapsto rz$ $r > 0$	$z \mapsto re^{i\theta} z$ $r \neq 1, \theta \neq 0$
action on $\mathbb{C}P^1 \cong S^2$	only one fixed point (all other points limit to this fixed point)	two fixed points, limit of any other orbit does not exist	two fixed points	

- **Finite order** in $PSL(2)(\mathbb{C}) \iff$ it is conjugate to f_λ with $\lambda^n = 1$ (root of unity)
  - $\implies$ elliptic
- Not all elliptic elements are of finite period, consider f_λ where $\lambda \in U(1)$ but not a root of unity.

action on distinct triples

> **Proposition:** The action
>
> $$PSL(2)(\mathbb{C}) \curvearrowright \boldsymbol{Ord}^3(S^2)$$
> $$(z_1, z_2, z_3) \overset{A}{\mapsto} (Az_1, Az_2, Az_3)$$
>
> is *transitive and faithful* as there is a **unique** Mobius transformation that maps $(\alpha, \beta, \gamma) \mapsto (1, 0, \infty)$ which is
>
> $$M_{\alpha,\beta,\gamma}(z) := \frac{\alpha - \gamma}{\alpha - \beta} \frac{z - \beta}{z - \gamma}$$
>
> Thus giving us a bijection
>
> $$PSL(2)(\mathbb{C}) \to \boldsymbol{Ord}^3(S^2)$$
>
> ?

Current note has 0 direct children and 0 total descendants.

- squishy
  - $PSL(2)(\mathbb{C}) \curvearrowright \mathbb{C}P^1 \cong_{\mathsf{Man}} S^2$

And it has 30 siblings.

- squishy
  - $SL(2)(\mathbb{R}) \curvearrowright \mathbb{R}^2$
  - $SL(2)(\mathbb{R}) \curvearrowright H^2_{\mathrm{U}}$
  - $\mathsf{Mat}_{\mathbb{C}}(2)$
  - $A_{r,}\text{\textunderscore}\langle z \mapsto \lambda z \rangle \backslash \mathbb{R}\boldsymbol{H}^2_{\mathrm{U}}$
  - $\mathbb{C} \setminus \{0\}\text{\textunderscore}\langle z \mapsto z+1 \rangle \backslash \mathbb{R}\boldsymbol{H}^2_{\mathrm{U}}$
  - $\mathbb{C} \setminus \{0, 1\}\text{\textunderscore}PSL(2)(\mathbb{Z})[2] \backslash \mathbb{R}\boldsymbol{H}^2_{\mathrm{U}}$
  - $\mathbb{R} \ltimes_\theta \mathbb{C}^2$
  - $\boldsymbol{Cl}(\mathbb{R}^n, \langle\ ,\ \rangle_{0,n})$
  - $\mathbb{C}^n$
  - $U(\mathbb{C} \setminus \{0\})$
  - $PSL(2)(\mathbb{C}) \curvearrowright \mathbb{C}P^1 \cong_{\mathsf{Man}} S^2$
  - $\mathbb{C}P^n$
  - Farey graph, complex and tree
  - GL
  - Heisenberg group
  - Incomplete hyperbolic polygon mod side-pairing
  - $\mathbb{R}\boldsymbol{H}^2\text{\textunderscore}\mathbb{R}H^2_{\mathrm{U}}\text{\textunderscore}D^2$
  - $\mathbb{R}\boldsymbol{H}^n$
  - D_n lattice
  - Matrices
  - $\mathsf{Mat}_{\mathbb{R}}(2)$
  - Mobius endomorphisms
  - Elementary group with finite and infinite sided Dirichlet polyhedra
  - Polynomials
  - $T^*S^1 \ncong_{\mathsf{Sym}} (\mathbb{R}^2 \setminus \{0\}, \mathrm{d}x \wedge \mathrm{d}y)$
  - $Z\left(\sum_{1 \le j \le n} z_j^2 - 1\right)(\mathbb{C}) \cong_{\mathsf{Sym}} T^*S^{n-1}$
  - SL
  - SO
  - Seifert–Weber dodecahedral 3-manifold
  - $U(n, \mathbb{C})$