Info

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$SL(2)(\mathbb{Z})$

Definition.

$$SL(2)(\mathbb{Z}) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$

$$\begin{cases} d = \frac{1+bc}{a} & a \neq 0 \\ & b \neq 0 \end{cases}$$

[1]

trace and eigenvalues

Given the trace $\mathrm{tr}(A) \in \mathbb{Z}$, the eigenvalues of $A \in SL(2)(\mathbb{Z})$ are

$$\frac{\mathrm{tr}(A)}{2} \pm \sqrt{\left(\frac{\mathrm{tr}(A)}{2}\right)^2 - 1}$$

based on the discriminant $\mathrm{tr}(A)^2 - 4$ we have these cases

$\mathrm{tr}(A)^2 - 4$	$\lvert\mathrm{tr}(A)\rvert$	**type**	**eigenvalues** λ, λ^{-1}
< 0	$0, 1$	elliptic	$\lambda, \lambda^{-1} \in \frac{\mathrm{tr}(A)}{2} \pm i\sqrt{1$ whose norm is 1, so $\lambda \in U(1) \setminus \{-1, 1\}$
0	2	parabolic	$\frac{\mathrm{tr}(A)}{2} \in \{-1, 1\}$
> 0	≥ 3	hyperbolic	$\lambda \in \mathbb{R} \setminus \{-1, 1\}$

operations

	inverse	**multiplying with** $-I$	**transpose**
matrix A	A^{-1}	$-A$	$A^{\intercal}$
eigenvalues λ, λ^{-1}			

generators

Proposition:

$$SL(2)(\mathbb{Z}) = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right\rangle$$

actions

elements, subgroups	**eigenvalues** λ, λ^{-1}	$SL(2)(\mathbb{R}) \curvearrowright$	$SL(2)(\mathbb{Z}) \curvearrowright$	$SL(2)(\mathbb{R}) \curvearrowright$ **by isometries**	$SL(2)(\mathbb{R})/Sl$	**on** $GL(2)(\mathbb{R})$
$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$			generator of Dehn twists			
$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$			Arnold cat map $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \curvearrowright T$	$\frac{2z+1}{z+1}$		
$\lvert\mathrm{tr}\rvert = 0, 1 \Leftarrow$ finite order	$\lambda \in U(1) \setminus \{$	periodic?	periodic	elliptic isometry (fixes one point)		
$\lvert\mathrm{tr}\rvert = 2$	$\lambda \in \{-1, 1\}$		reducible; infinite order but not hyperbolic	parabolic isometry (fixes one point in boundary)		
$\lvert\mathrm{tr}\rvert \geq 3$	$\lambda \in \mathbb{R} \setminus \{-1$	hyperbolic	(pseudo-)Anosov	hyperbolic isometry (fixes two points in boundary)		
"squishy.SL. Z 2 sub Sanov#^b mrihn" could not be found.		plays ping pong!				
$\Gamma[N]$						
Γ modular						
Γ torsion-free				quotients are higher genus surfaces		

Proposition:

elements, subgroups	**eigenvalues** λ, λ^{-1}	$SL(2)(\mathbb{R}) \curvearrowright \mathbb{R}^2$	$SL(2)(\mathbb{Z}) \curvearrowright T^2$	$SL(2)(\mathbb{R}) \curvearrowright \mathbb{R}H^2$ **by isometries**
$\lvert \mathrm{tr} \rvert = 0, 1 \iff$ finite order	$\lambda \in U(1) \setminus \{-1, 1\}$	periodic?	periodic	elliptic isometry (fixes one point)
$\lvert \mathrm{tr} \rvert = 2$	$\lambda \in \{-1, 1\}$		reducible; infinite order but not hyperbolic	parabolic isometry (fixes one point in boundary)
$\lvert \mathrm{tr} \rvert \geq 3$	$\lambda \in \mathbb{R} \setminus \{-1, 1\}$	hyperbolic	(pseudo-)Anosov	hyperbolic isometry (fixes two points in boundary)

[2]

Current note has 0 direct children and 0 total descendants.

- squishy
  - SL
    - $SL(2)(\mathbb{R})$
    - $SL(2)(\mathbb{Z})$
- stretchy
  - $GL(2)(\mathbb{Z}) \curvearrowright T^2$
    - $SL(2)(\mathbb{Z}) \curvearrowright T^2$

And it has 15 siblings.

- squishy
  - SL
    - SL(2)
    - $SL(2)(\mathbb{C})$
    - $PSL(n, \mathbb{C})$
    - $SL(2)(\mathbb{R})$
    - $SL(2)(\mathbb{R}) / SL(2)(\mathbb{Z})$
    - H_q
    - $PSL(2)(\mathbb{Z})[2]$
    - $[PSL(2)(\mathbb{Z})[2], PSL(2)(\mathbb{Z})[2$]]
    - $PSL(2)(\mathbb{Z})[N]$
    - $SL(3, \mathbb{R})$
    - $SL(2)(\mathbb{Z})$
    - $PSL(2)(\mathbb{Z})$
    - Congruent subgroups of $PSL(2)(\mathbb{Z})$
    - $SL(n, \mathbb{Z})$
    - $SL(2)(\mathbb{Z}_2)$

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1. kconrad.math.uconn.edu/blurbs/grouptheory/SL(2,Z).pdf ↩
2. Congruence Subgroups ↩