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# Homotopy invariance of cohomology of differential forms

**(Homotopy invariance of $H^k(\Omega^\bullet)$)** Given two continuously homotopic, smooth maps

$$f, g : M \to N$$

between manifolds possibly with boundary, we can always produce a smooth homotopy between them. Two such smoothly homotopic maps induce equal homomorphisms in the cohomology of differential forms

$$f^* = g^* : \Omega^\bullet(N) \to \Omega^\bullet(M)$$

In particular, if

$$\alpha : M \to N, \beta : N \to M$$

is a smooth homotopy equivalence, then $\alpha^*, \beta^*$ induce isomorphisms in the cohomology of differential forms.

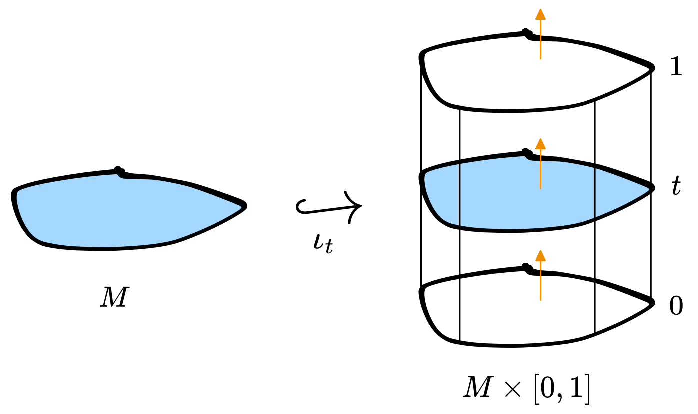
## the homotopy operator is an integral

Definition. Homotopy operator from forms on $M \times I$ to forms on $M$

Let $M$ be a smooth manifold with or without boundary and consider the map

$$\begin{aligned} \iota_t : M &\to M \times [0,1] \\ x &\mapsto (x,t) \end{aligned}$$

for a fixed $t \in [0,1]$.




Let $(-, s) \in M \times [0,1]$ be the coordinate. We define

$$\begin{aligned} h : \Omega^k(M \times I) &\to \Omega^{k-1}(M) \\ \omega &\mapsto h\omega \\ (h\omega)_q &:= \int_{t \in [0,1]} \iota_t^*(\iota_{\frac{\partial}{\partial s}}(\omega_{q,t})) \\ (h\omega)_q(-) &= \int_{t\in[0,1]} \omega(\frac{\partial}{\partial s}, \mathrm{d}\iota_t(-)) \end{aligned}$$

> Proposition:

$$(\iota_1^* - \iota_0^*)(\omega) = h(\mathrm{d}\omega) + \mathrm{d}(h\omega)$$

- Because $\iota_t = \exp(t\frac{\partial}{\partial s}) \circ \iota_0$ we have

$$\begin{aligned} \iota_t^*\omega &= \iota_0^*(\exp(t\frac{\partial}{\partial s})^*\omega) \\ \frac{\mathrm{d}}{\mathrm{d}t}\iota_t^*\omega &= \frac{\mathrm{d}}{\mathrm{d}t}\iota_0^*(\exp(t\frac{\partial}{\partial s})^*\omega) \\ &= \iota_0^*(\frac{\mathrm{d}}{\mathrm{d}t}\exp(t\frac{\partial}{\partial s})^*\omega) \\ &= \iota_0^*(\exp(t\frac{\partial}{\partial s})^*(\mathfrak{L}_{\frac{\partial}{\partial s}}\omega)) \\ &= \iota_t^*(\mathfrak{L}_{\frac{\partial}{\partial s}}\omega) \end{aligned}$$

  thus by integrating both sides on $t \in [0,1]$

$$(\iota_1^* - \iota_0^*)(\omega) = \int_{t\in[0,1]} \iota_t^*(\mathfrak{L}_{\frac{\partial}{\partial s}}\omega)$$

- Now, we also have

$$h(\mathrm{d}\omega) + \mathrm{d}(h\omega) = \int_{[0,1]} \iota_t^*(\mathfrak{L}_{\frac{\partial}{\partial s}}\omega)$$

## homotopy invariance of the cohomology

Given a smooth homotopy

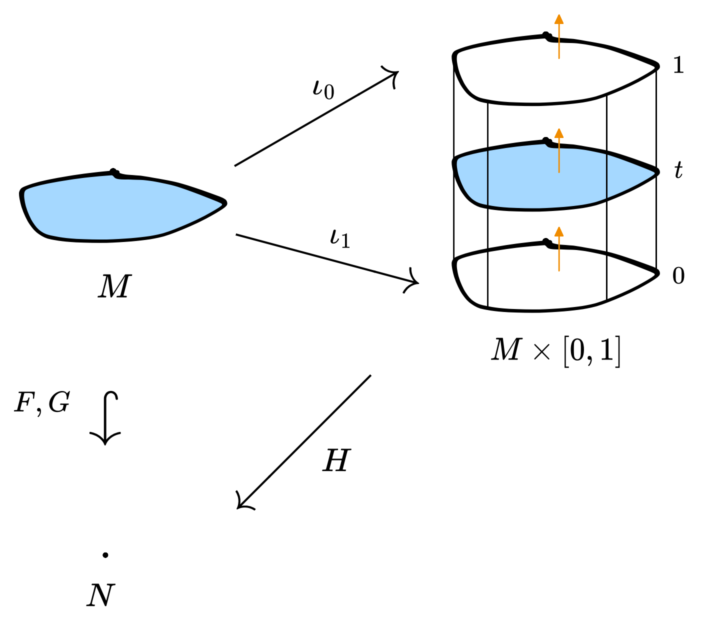
$$H : M \times I \to N$$

between the smooth maps

$$F, G : M \to N$$

we write

$$\begin{aligned} F &= H(,0) = H \circ \iota_0 \\ G &= H(,1) = G \circ \iota_1 \end{aligned}$$




then we have

$$
\begin{aligned}
F^* &= \iota_0^* \circ H^* \\
G^* &= \iota_1 \circ H^*
\end{aligned}
$$

and because

$$\iota_0 = \iota_1 : H^k(\Omega^\bullet(N)) \to H^k(\Omega^\bullet(M))$$

we have

$$F^* = G^* : H^k(\Omega^\bullet(N)) \to H^k(\Omega^\bullet(M))$$

Thus homotopy equivalence produces an isomorphism in the cohomology.

## consequences for contractible manifolds

For a point manifold $\{p\}$, the de Rham cohomology

$$H^k_{dR}(\{p\}) = \begin{cases} \mathbb{R} & k = 0 \\ 0 & k > 0 \end{cases}$$

The *Poincare lemma* for contractible sets:

The de Rham cohomology of a contractible set $U$ in a smooth manifold are

$$H^k_{dR}(U) = \begin{cases} \mathbb{R} & k = 0 \\ 0 & k > 0 \end{cases}$$

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Current note has 0 direct children and 0 total descendants.

- structures based on sets
  - manifolds
    - smooth $\mathbb{R}$-manifold
      - Tangents on a smooth manifold
        - Differential forms on smooth manifolds
          - Chain complex of differential forms on a smooth manifold
            - Homotopy invariance of cohomology of differential forms

And it has 1 siblings.


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