Info

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G-bundles over smooth manifolds

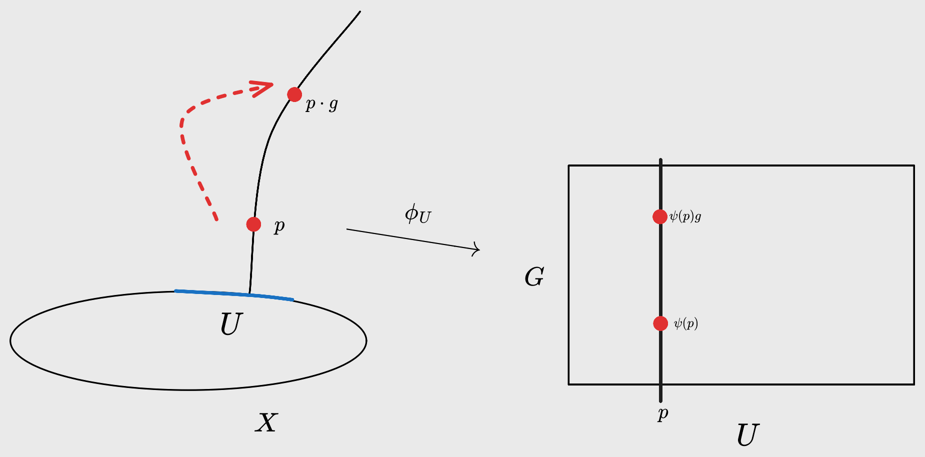
$$G \curvearrowright P \to X$$

Definition. G-bundles over smooth manifolds $G \curvearrowright P \to M$

A **smooth principal G-bundle over M**

$$G \curvearrowright P \to M := (\Phi : G \curvearrowright P, \pi_P : P \to M, \mathscr{U})$$

for a Lie group G and a smooth manifolds P, M is a

> **Definition.** Principal G-bundles $G \curvearrowright P \to X$
>
> A **topological principal G-bundle over X**
>
> $$G \curvearrowright P \to X := (G \curvearrowright P, \pi_P : P \to X, \mathscr{U})$$
>
> where
>
> - G is a Topological group called the structure group
> - X is a T_2 topological space (Hausdorff?)
> - a surjective continuous map
>
> $$\pi_P : P \to X$$
>
> - a right group action
>
> $$G \curvearrowright P$$
>
>   - preserves fibres
> - *locally trivial* for each $x \in X$, $\exists (U, \phi_U) \in \mathscr{U}$ where
>   - U is a open nbd of x in X
>   - and a homeomorphism
>
> $$\begin{aligned} \phi_U : \pi_P^{-1}(U) &\to U \times G \\ p &\mapsto (\pi_P(p), \psi) \end{aligned}$$
>
>   such that
>
> $$\psi(p \cdot g) = \psi(p)g$$
>
> 
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where the bundle maps, the group action and local trivializations are smooth.

The derivative of the group action

$$\Phi : G \to \mathrm{Diff}(P)$$

gives

$$\mathrm{d}_i\Phi : \mathfrak{g} \to \mathrm{Vec}(P)$$

local trivializations

Definition. From the data $G \curvearrowright P \to M$ we can define transition functions

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$$

such that

$$\psi_\alpha(p) = g_{\alpha\beta}(\pi(p))\psi_\beta(p)$$

Transition functions of a local trivialization satisfy the **cocycle condition**

$$g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma}$$

on $U_\alpha \cap U_\beta \cap U_\beta$. Moreover $g_{\alpha\gamma} = i_G$ and $g_{\beta\alpha} = g_{\alpha\beta}^{-1}$

automorphisms

- **automorphism of a smooth G-bundle** $G \curvearrowright P \to M$
- **local gauge transformation** are maps

$$g_\alpha : U_\alpha \to G$$

on a trivializing open set $U_\alpha \in \mathscr{U}$

  - define the **local bundle isomorphism** by $\varphi_\alpha(p) := pg_\alpha(\pi(p))$

	automorphism of a smooth G-bundle $G \curvearrowright P \to M$	sections of the adjoint bundle $\mathrm{Ad}(P) \to M$	maps to G
local			$g_\alpha : U_\alpha \to G$
converting $\longrightarrow$			
converting $\longleftarrow$			
at a point $x \in M$			an element of G
global			

connections

Definition. Connection on a smooth G-bundle

A connection on a smooth G-bundle $G \curvearrowright P \to X$ is a smooth $\mathfrak{g}$-valued 1-form on P

$$\omega \in \Gamma(\Lambda^1 T^* P \otimes \mathfrak{g})$$

such that

- $$(\Phi^g)^* \omega = \mathrm{Ad}_{g^{-1}} \circ \omega$$
- $$\omega(A^\#) = A$$

lifting Lie algebra valued 1-forms on base to connection forms on the bundle

On G-bundle $(G \curvearrowright P \xrightarrow{\pi_P} X)$ with trivialization (V_i, ϕ_i) let we have a family of Lie algebra valued differential forms $A_j \in \mathrm{Forms}^1(V_j, \mathfrak{g})$ with

$$A_j = \mathrm{Ad}_{g_{ij}^{-1}} A_i + g_{ij}^* \Theta \text{ on } V_j \cap V_i$$

where $g_{ij} : V_i \cap V_j \to G$ is the transition function. Then there exists a unique connection $\omega \in \mathrm{Forms}^1(P, \mathfrak{g})$ of the G-bundle such that

$$A_j = s_j^* \omega$$

where $s_j : V_j \to \pi_P^{-1}(V_j)$ is the canonical section associated with the trivialization.

horizontal lift of curves

On G-bundle with connection $(G \curvearrowright P \xrightarrow{\pi_P} X, \omega)$, for every *smooth curve* and choice of starting point

$$\alpha : [0,1] \to X$$
$$p \in \pi_P^{-1}(\alpha(0))$$

there exists a smooth curve on P which is a **lift of α starting at p**

$$\tilde{\alpha}_p : [0,1] \to P$$
$$\tilde{\alpha}_p(0) = p$$
$$\pi_P \circ \tilde{\alpha}_p = \alpha$$

and whose velocity is horizontal

$$t \in [0,1] \implies \tilde{\alpha}_p'(t) \in \mathrm{Hor}_{\tilde{\alpha}_p(t)}(P)$$

Let U_i be a chart which contains γ and take a section

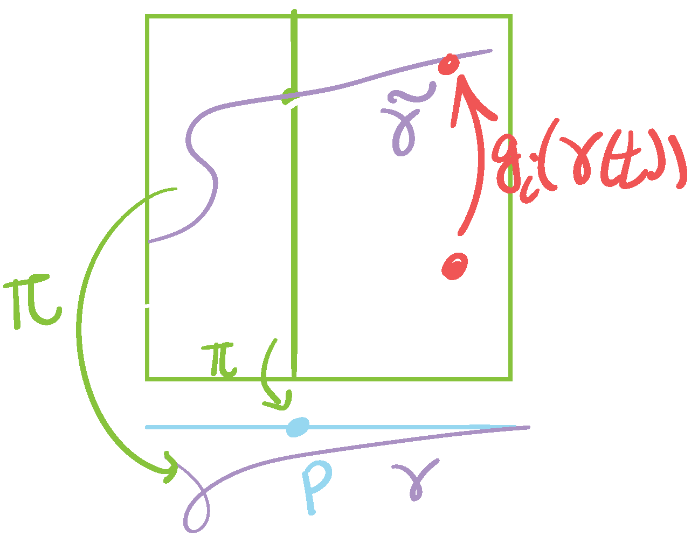
$$\sigma_i : U_i \to \pi^{-1}(U_i)$$

The horizontal lift may look like

$$\tilde{\gamma}(t) = \sigma_i(\gamma(t)) g_i(\gamma(t))$$

where we may choose σ_i and $g_i : U_i \to G$ to be such that

$$\sigma_i(\gamma(0)) = \tilde{\gamma}(0), g_i(\tilde{\gamma}(0)) = i_G$$



The condition on $g_i(\gamma(t))$ is

$$\frac{\mathrm{d}}{\mathrm{d}t} g_i(t) = -\omega(\sigma_{i*} X) g_i(t)$$

whose unique solution is

$$g_i(t) = \mathrm{pexp}\left(i \int_{\gamma|[0,t]} \mathcal{A}_{i\mu} \mathrm{d}x^\mu\right)$$

Current note has 0 direct children and 0 total descendants.

- structures based on sets
  - manifolds
    - smooth $\mathbb{R}$-manifold
      - G-bundles over smooth manifolds $G \curvearrowright P \to X$

And it has 19 siblings.

- structures based on sets
  - manifolds
    - smooth $\mathbb{R}$-manifold
      - Smooth Lie group action on a smooth manifold
      - Borel measures on a smooth manifold
      - G-bundles over smooth manifolds $G \curvearrowright P \to X$
      - Vector bundle on smooth manifolds
      - Curves in smooth manifolds
      - Density on smooth manifolds
      - Functions between smooth manifolds
      - Functions from smooth manifolds to $\mathbb{R}$
      - Flows on a smooth manifold
      - Locally $G \curvearrowright X$-manifolds
      - Smooth singular homology of a smooth manifold
      - Isomorphisms and automorphisms of smooth manifolds
      - Lagrangian flows
      - Local transformation rules
      - Moduli space of Riemannian metrics on a smooth manifold
      - Moduli space of quotients
      - Quotient manifolds
      - Submanifolds of smooth manifolds
      - Tangents on a smooth manifold